Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHEMATIK

NEUE UNGLEICHUNGEN FÜR DIE MISCHVOLUMEN KONVEXER KÖRPER

Von A. ALEXANDROV

(Vorgelegt von I. M. Vinogradow, Mitglied der Akademie, d. 10. XI. 1936)

Es seien im n-dimensionalen Raume N Richtungen $\overline{n}_1, \ldots, \overline{n}_N$ derart gegeben, dass die zu ihnen senkrechte Ebenen $P_1, \ldots, P_N$ ein konvexes Vielflach Π begrenzen. Die Ebene P_i bezeichnen wir als wesentlich Begrenzende, wenn sie auf Π eine Fläche von nicht-nulltem $(n-1)$-dimensionalem Volumen gibt. Zwei Vielecke, die ein und dasselbe Normalensystem $\overline{n}_1, \ldots, \overline{n}_N$ haben, werden wir als analoge bezeichnen, wenn die Gerade P_i für das eine sowohl wie für das andere wesentlich Begrenzende ist. Zwei Vielflache heissen analog, wenn sie das gleiche Normalensystem besitzen und alle ihre Flächen von nicht-nulltem $(n-1)$-dimensionalem Volumen sind und deren entsprechende Flächen analog sind. Die Abstände H_i vom Ursprung O bis zu den begrenzenden Flächen P_i nennen wir Stützzahlen des Vielflaches. Wenn wir als Ursprung in P_i die Projektion des Ursprungs O auf dieselbe nehmen, dann ist der Abstand von demselben bis zum Schnitt von P_i mit P_k (Θ_{ik} bedeutet der Winkel zwischen $\overline{n}_i$ und $\overline{n}_k$):

$$H_{ik} = \frac{H_k - H_i \cos \Theta_{ik}}{\sin \Theta_{ik}} . \qquad (1)$$

Das sind die Stützzahlen der i-ten Fläche.

Der Mischung der analogen Vielflache entspricht das Summieren ihrer Stützzahlen. Für die Vielecke ist diese Behauptung offenbar. Aus (1) ist klar, dass bei dem Summieren der Stützzahlen der Vielflache sich auch die Stützzahlen ihrer Flächen summieren. Ausserdem ist bekannt, dass die Fläche eines gemischten Vielflaches das Resultat der Mischung paralleler Flächen ist. Dabei ist unsere Behauptung, wenn sie für $(n-1)$-dimensionale Vielflache richtig ist, auch für n-dimensionale richtig.

Es sei V das Volumen eines Vielflachs, das durch Mischung analoger Vielflache $H^{(1)}, \ldots, H^{(n)}$ erhalten worden ist, und F_i — die Volumen seiner Flächen.

Dann erhalten wir, wenn wir mit $V(H^{(j_1)} \ldots H^{(j_n)})$ die Mischvolumen bezeichnen:

$$V = \sum_{j_1 \ldots j_n} \lambda_{j_1} \ldots \lambda_{j_n} V(H^{(j_1)} \ldots H^{(j_n)}) = \frac{1}{n} \sum_{i=1}^{N} \Big(\sum_{k=1}^{n} \lambda_k H_i^{(k)} \Big) F_i , \qquad (2)$$

und da seinerseits

$$F_i = \sum_{j_1 \dots j_{n-1}} \lambda_{j_1} \dots \lambda_{j_{n-1}} F_i (H^{(j_1)} \dots H^{(j_{n-1})}), \tag{3}$$

$$V = \sum_{j_1 \dots j_n} \lambda_{j_1} \dots \lambda_{j_n} \sum_{i=1}^{N} H_i^{(j_n)} F_i (H^{(j_1)} \dots H^{(j_{n-1})}). \tag{4}$$

Wir zeigen, dass unabhängig von der Reihenfolge der Indices $j_1 \dots j_n$:

$$V (H^{(j_1)} \dots H^{(j_n)}) = \frac{1}{n} \sum_{i=1}^{N} H_i^{(j_n)} F_i (H^{(j_1)} \dots H^{(j_{n-1})}). \tag{5}$$

Wir nehmen an, dass dies zutrifft für den $(n-1)$-dimensionalen Raum. Dann ist

$$F_i (H^{(j_1)} \dots H^{(j_{n-1})}) = \frac{1}{n-1} \sum_{k=1}^{N} H_{ik}^{(j_{n-1})} F_{ik} (H^{(j_1)} \dots H^{(j_{n-2})}), \tag{6}$$

$$F_i (H^{(j_1)} \dots H^{(j_n)}) = \frac{1}{n-1} \sum_{k=1}^{N} H_{ik}^{(j_n)} F_{ik} (H^{(j_1)} \dots H^{(j_{n-1})}). \tag{7}$$

Indem wir hier die Werte H_{ik} aus (1) einsetzen, (6) mit $H_i^{(j_n)}$ und (7) mit $H_i^{(j_{n-1})}$ multiplizieren, und über i summieren, erhalten wir:

$$\sum_{i=1}^{N} H_i^{(j_n)} F_i^2 (H^{(j_1)} \dots H^{(j_{n-1})}) = \sum_{i=1}^{N} H_i^{(j_{n-1})} F_i (H^{(j_1)} \dots H^{(j_n)}). \tag{8}$$

Hieraus ergibt sich im Zusammenhang mit Formel (4) Formel (5). Man kann leicht zeigen, dass der Beweis für die Ungleichung:

$$V (H^{(1)} \dots H^{(n-1)} H^{(n)})^2 \geqslant V (H^{(1)} \dots H^{n-1} H^{n-1}) V (H^{(1)} \dots H^{(n)} H^{(n)}), \tag{9}$$

gleichwertig (im Falle der analogen Vielflache) ist dem Beweis, dass unter der Bedingung:

$$V (H^{(1)} \dots H^{(n-1)} Z) = 0, \tag{10}$$

die quadratische Form der Veränderlichen Z_i:

$$V (H^{(1)} \dots H^{(n-2)} ZZ) \leqslant 0, \tag{11}$$

wo das Gleichheitszeichen nur bei einer linearen Stützfunktion Z steht.

Die Überführung der Form (10) in die kanonische Form unter der Bedingung:

$$\sum_{i=1}^{N} Z_i^2 \frac{F_i (H^{(1)} \dots H^{(n-1)})}{H_i^{(n-1)}} = 1 \tag{12}$$

führt zur Lösung der Gleichungen:

$$F_i (H^{(1)} \dots H^{(n-2)} Z) = \lambda \frac{F_i (H^{(1)} \dots H^{(n-1)})}{H_i^{(n-1)}} Z_i. \tag{13}$$

Die Lösung für $\lambda = 1$, $Z = H^{(n-1)}$ spielt wegen der Bedingung (10) keine Rolle. Für $\lambda = 0$, wenn $Z = H - H^{(n-2)}$, wo H ein analoges Vielflach ist, erhalten wir:

$$F_i(H^{(1)} \dots H^{(n-2)} H) = F_i(H^{(1)} \dots H^{(n-2)} H^{(n-2)}). \tag{14}$$

Unser Theorem führt für den Fall von Vielecken zum Brunn-Minkowskischen Theorem. Ist das Theorem richtig für $(n-1)$-dimensionale Vielflache, so erhalten wir:

$$F_i H^{(1)} \dots H^{(n-2)} H) \geqslant F_i(H^{(1)} \dots HH), \tag{15}$$

oder

$$F_i(H^{(1)} \dots H^{(n-3)} ZZ) \leqslant 0. \tag{16}$$

Aus (8) erhalten wir, dass

$$\sum_{i=1}^{N} H_i^{(n-2)} F_i(H^{(1)} \dots ZZ) = \sum_{i=1}^{N} Z_i F_i(H^{(1)} \dots H^{(n-2)} Z) = 0, \tag{17}$$

wegen (13) für $\lambda = 0$, und da $H_i^{(n-1)} > 0$, so ist nach (16):

$$F_i(H^{(1)} \dots H^{(n-3)} ZZ) = 0. \tag{18}$$

Folglich ist Z eine lineare Stützfunktion, so dass das Gleichheitszeichen in (11) unter der Bedingung (10) nur für ein solches Z möglich ist.

Für $H^{(1)} = \dots = H^{(n-2)}$ stellt die Ungleichung (9) die bekannte Minkowskische Ungleichung dar. Daher ist in diesem Falle $\lambda = 1$ der einzige positive λ-Wert. Aber wir haben gezeigt, dass $\lambda = 0$ immer denselben Lösungen von (13) entspricht.

Daher sehen wir, wenn wir stetig von gleichen $H^{(1)}, \dots, H^{(n-2)}$ zu verschiedenen übergehen, dass dies auch für den allgemeinen Fall richtig ist, woraus auch unser Theorem folgt.

Die Übertragung der Ungleichung (9) auf beliebige konvexe Körper gewinnt man durch folgendes Theorem: Man kann beliebige n nichtentartete konwexe Körper mit erwünschter Genauigkeit durch analoge Vielflache approximieren. Die Erbringung des Beweises dieser einfachen Satzes unterlassen wir [1].

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Eingegangen
d. 10. XI. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHÉMATIQUES

Par B. GNEDENKO

SUR L'UNICITÉ DU SYSTÈME DE FONCTIONS ORTHOGONALES INVARIANT PAR RAPPORT À LA DÉRIVATION

(Présenté par S. N. Bernstein, de l'Académie, le 13. XII. 1936)

En 1929 B. Gagaeff (1) a donné la solution du problème suivant, qui a été posé par N. Lusin (2): «La dérivation et l'intégration du système de fonctions trigonométriques:

$$\{\cos 2\pi nx,\ \sin 2\pi nx, \qquad n=1, 2, 3, \ldots,\ 0\leqslant x\leqslant 1,\} \qquad (1)$$

nous ramène (abstraction faite des facteurs) au même système de fonctions. Il y a intérêt de savoir, s'il existe un a u t r e système de fonctions orthogonales jouissant de la même propriété».

Gagaeff affirme qu'il n'existe qu'un seul système—celui des fonctions trigonométriques (1) qui possède la propriété précédente.

Or, je veux indiquer ici que Gagaeff n'a pas remarqué un autre système de fonctions continues, à savoir:

$$\left\{\cos 2\pi\left(n-\tfrac{1}{2}\right)x,\ \sin 2\pi\left(n-\tfrac{1}{2}\right)x; \qquad n=1, 2, 3, \ldots,\ 0\leqslant x\leqslant 1\right\}. \qquad (2)$$

Plus précisément, il faudrait dire que tous les systèmes de fonctions orthogonales donnant la solution du problème de Lusin (pour l'intervalle $0\leqslant x\leqslant 1$) sont (à facteurs constants près) ou bien de la forme:

$$\{\cos(2\pi n_k x+\psi_k),\ \sin(2\pi n_k x+\psi_k); \qquad 0\leqslant x\leqslant 1\}, \qquad (3)$$

ou bien de la forme:

$$\left\{\cos\left[2\pi\left(n_k-\tfrac{1}{2}\right)x+\psi_k\right],\ \sin\left[2\pi\left(n_k-\tfrac{1}{2}\right)x+\psi_k\right];\ \ 0\leqslant x\leqslant 1\right\}, \qquad (4)$$

$\{n_k\}$ étant une suite arbitraire de nombres entiers positifs différents deux à deux et ψ_k — des constantes arbitraires. Si l'on veut se borner à la recherche des systèmes complets (fermés), il suffit de prendre, au lieu des suites $\{n_k\}$ arbitraires, la suite complète des nombres entiers positifs (l e s y s t è m e (2) e s t c o m p l e t).

En effet, Gagaeff a démontré dans la note citée que la forme

générale des systèmes de fonctions orthogonales invariant par rapport à la dérivation est

$$\{\varphi_n(x) = \rho_n \cos(\lambda_n x + \psi_n)\}, \tag{5}$$

où ρ_n, λ_n et ψ_n sont des constantes.

Dans ces conditions, la fonction

$$\frac{1}{\lambda_n}\varphi_n'(x) = -\sin(\lambda_n x + \psi_n)$$

est de même une fonction de se système (à un facteur constant près). En vertu de l'égalité

$$\int_0^1 \varphi_n(x)\,\varphi_n'(x)\,dx = [\cos^2(\lambda_n + \psi_n) - \cos^2\psi_n] = 0$$

on voit que pour λ_n deux cas seulement sont possibles:

$$\lambda_n = \pi k \text{ et } \lambda_n = \pi k - 2\psi_n,\ k\text{—un entier.}$$

Il est évident que $\lambda_n \neq \lambda_m$, si $\varphi_m(x)$ est différent de $\varphi_n(x)$ et de $\varphi_n'(x)$.

Nous envisageons séparément tous les cas qui peuvent se présenter.

1-r cas $\lambda_n = \pi k$ et $\lambda_m = \pi r$, k et r sont des entiers.

Comme, d'une part, les combinaisons linéaires de $\varphi_n(x)$ et $\varphi_n'(x)$ et, d'autre part, les combinaisons linéaires de $\varphi_m(x)$ et $\varphi_m'(x)$ sont orthogonales, nous avons:

$$\int_0^1 \cos k\pi x \cdot \sin r\pi x\,dx = -\frac{1}{r+k}[1 - \cos(r+k)\pi] -$$

$$-\frac{1}{r-k}[1 - \cos(r-k)\pi] = 0.$$

Mais cette relation n'est possible que dans le cas où r et k sont pairs ou impairs simultanément (on peut supposer sans restreindre la généralité que $r > k$). Nous avons donc:

ou bien tous les k sont de la forme $2s+1$,

ou bien ils sont de la forme $k = 2s$,

ou bien les s sont des entiers.

2-me cas. $\lambda_n = \pi k - 2\psi_n$, $\lambda_m = \pi r$.

Nous pouvons supposer que $2\psi_n$ n'est égal ni à $\pi(k+r)$ ni à $\pi(k-r)$, car dans le cas contraire nous sommes dans les conditions du cas précédent.

Sans restreindre la généralité on peut supposer que $\pi(k+r) - 2\psi_n$ et $\pi(k-r) - 2\psi_n$ ont le même signe.

Nous construisons, comme dans le 1-r cas, deux fonctions orthogonales $\cos(\pi k - 2\psi_n)x$ et $\sin \pi r x$. L'égalité:

$$\int_0^1 \cos(\pi k - 2\psi_n)x \cdot \sin \pi r x\,dx =$$

$$= \frac{\{1 - \cos[\pi(k+r) - 2\psi_n]\}}{\pi(k+r) - 2\psi_n} + \frac{\{1 - \cos[\pi(k-r) - 2\psi_n]\}}{\pi(k-r) - 2\psi_n} = 0,$$

nous montre que l'on a $\pi(k+r) - 2\phi_n = 2M\pi$ et $\pi(k-r) 2\phi_n = 2N\pi$, où M et N sont des entiers.

Par conséquent, comme dans le cas précédent, nous avons en effet $\lambda_n = \pi p$ et $\lambda_m = \pi q$, où p et q sont des entiers.

3-me cas. $\lambda_n = \pi k - 2\phi_n$, $\lambda_m = \pi r - 2\phi_m$.

Nous pouvons supposer que les quantités:

$$n(k+r) - 2\phi_n - 2\phi_m \text{ et } \pi(k-r) - 2\phi_n + 2\phi_m$$

ont le même signe, car dans le cas contraire les quantités suivantes auraient le même signe:

$$\pi(k+r) - 2\phi_n - 2\phi_m \text{ et } \pi(r-k) - 2\phi_m + 2\phi_n.$$

L'égalité $\pi(k+r) - 2\phi_n - 2\phi_m = 0$ ainsi que l'égalité $\pi(k-r) - -2\phi_n + \phi_m = 0$ est impossible parce que les fonctions $\varphi_n(x)$, $\varphi_n'(x)$, $\varphi_m(x)$, $\varphi_m'(x)$ ne sont pas linéairement indépendantes.

Comme dans les cas précédents nous construisons deux nouvelles fonctions orthogonales $\cos(\pi k - 2\phi_n)x$ et $\sin(\pi r - 2\phi_m)x$.

L'égalité:

$$\int_0^1 \cos(\pi k - 2\phi_n)x \cdot \sin(\pi r - 2\phi_m)x\,dx =$$

$$= \frac{\left\{1 - \cos[\pi(k+r) - 2\phi_n - 2\phi_m]\right\}_0^1}{\pi(k+r) - 2\phi_n - 2\phi_m} = \frac{\left\{1 - \cos[\pi(k-r) - 2\phi_n + 2\phi_m]\right\}}{\pi(k-r) - 2\phi_n + 2\phi_m} = 0,$$

nous montre qu'on doit avoir nécessairement

$$\pi(k+r) - 2\phi_n - 2\phi_m = 2M\pi$$

et

$$\pi(k-r) - 2\phi_n + 2\phi_m = 2N\pi$$

où M et N sont des entiers.

Ces équations nous montrent que $2\phi_n = s\pi$ et $2\phi_m = t\pi$, où s et t sont des entiers.

Donc il ne reste que deux cas possibles:

ou bien tous les λ_n sont de la forme $2\pi k$,

ou bien tous les λ_n sont de la forme $2\pi\left(k - \frac{1}{2}\right)$.

La méthode de Gagaeff permet de donner une réponse aux questions analogues à celle de N. Lusin:

1. Trouver tous les systèmes de fonctions orthogonales tels que la derivée d'une fonction quelconque du système appartient elle-même a ce système.

2. Trouver tous les systèmes de fonctions orthogonales qui sont compris dans le système des dérivées des fonctions du même système.

Les seuls systèmes de fonctions donnant la réponse à ces questions sont encore (3) et (4).

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Manuscrit reçu
le 13. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHÉMATIQUES

UNE MÉTHODE D'INTÉGRATION DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU PREMIER ORDRE, LINÉAIRES PAR RAPPORT AUX JACOBIENS ET A PLUSIEURS FONCTIONS INCONNUES

Par M. KOURENSKY

(Présenté par A. N. Krylov, de l'Académie, le 14. XII. 1936)

1. Pour les équations

$$u_1(x,\ y,\ z,\ z') = C_1;\quad u_2(x,\ y,\ z,\ z') = C_2$$

où z et z' sont des fonctions de x et y, l'élimination de la fonction arbitraire φ de

$$u_1(x,\ y,\ z,\ z') = \varphi[u_2(x,\ y,\ z,\ z')] \tag{1}$$

conduit à l'équation linéaire par rapport aux jacobiens de la forme:

$$Ap + Bq + C + A'p' + B'q' + D(pq' - qp') = 0, \tag{2}$$

dont les coefficients vérifient la relation $AB' - BA' = CD$, nommée condition de Hamburger. L'intégrale générale (1) de l'équation (2) est nommée intégrale de Hamburger; un cas particulier de cette intégrale représente l'intégrale générale dans la théorie classique de l'intégration des équations linéaires à une seule fonction inconnue z.

Suivant la formule:

$$u_1 + \alpha n_2 + \beta = \gamma\varphi(n_2 + \delta u_1 + \varepsilon),$$

où $\alpha, \ldots, \varepsilon$, sont des fonctions de $x, \ldots, z'$, nous obtiendrons une équation nouvelle, linéaire par rapport aux jacobiens, dont les coefficients nouveaux vérifient la condition mentionée; pour $\gamma = 1$, $\alpha = \beta = \delta = \varepsilon = 0$ nous obtiendrons l'intégrale générale de la première équation.

Nous obtiendrons les conclusions analogues pour les intégrales d'équations linéaires par rapport aux jacobiens dans le cas de trois ou de plusieurs fonctions et arguments:

$$u_1(x_1,\ x_2,\ x_3,\ z,\ z') = \varphi[u_2(x_1,\ \ldots,\ z');\ u_3(x_1,\ \ldots,\ z']$$
$$u_1(x,\ y,\ z,\ z',\ z'') = \varphi[u_2(x,\ \ldots,\ z'')];\ \ldots$$

2. Introduisons les notations

$$\frac{\partial(u_1, u_2)}{\partial(x, y)} = (xy); \quad \frac{\partial(u_1, u_2)}{\partial(x, z)} = (xz); \quad \ldots; \quad \frac{\partial(u_1, u_2)}{\partial(z, z')} = (zz').$$

La dérivation des équations:

$$u_1 = C_1, \quad u_2 = C_2,$$

conduit aux relations

$$(zz')\,p + (xz') = 0; \quad (zz')\,q + (yz') = 0; \quad \ldots; \quad (zz')\,(pq' - qp') - (xy) = 0,$$

et nous avons au lieu de (2):

$$(xz')\,A + (yz')\,B - (zz')\,C + (zx)\,A' + (zy)\,B' - (xy)\,D = 0. \tag{3}$$

D'autre part l'élimination de dz' des différentielles totales

$$\frac{\partial u_i}{\partial x}\,dx + \frac{\partial u_i}{\partial y}\,dy + \frac{\partial u_i}{\partial z}\,dz + \frac{\partial u_i}{\partial z'}\,dz' = 0 \qquad (i = 1, 2)$$

donne

$$(xz')\,dx + (yz')\,dy + (zz')\,dz = 0. \tag{4}$$

Donc, quand nous aurons une fonction $u_1(x, y, z, z')$, telle que $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{-C}$ alors pour chaque fonction $u_2(x, y, z, z')$ l'équation (4) sera équivalente à la suivante:

$$(xz')\,A + (yz')\,B - (zz')\,C = 0,$$

et, par conséquant, l'équation (3) se réduit à la suivante:

$$(zx)\,A' + (zy)\,B' - (xy)\,D = 0.$$

Cela veut dire que, quand nous aurons déterminé la fonction u_1 comme une intégrale particulière de l'équation linéaire à une seule fonction inconnue:

$$A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} - C\frac{\partial f}{\partial z} = 0, \tag{5}$$

nous aurons u_2 comme une intégrale particulière de l'équation linéaire à une seule fonction:

$$(A'n_{13} + Du_{12})\frac{\partial u}{\partial x} + B'u_{13} - Du_{11})\frac{\partial u}{\partial y} - (A'u_{11} + B'u_{12})\frac{\partial u}{\partial z} = 0, \tag{6}$$

où

$$u_{11} = \frac{\partial u_1}{\partial x}, \quad u_{12} = \frac{\partial u_1}{\partial y}, \quad u_{13} = \frac{\partial u_1}{\partial z}.$$

Dans les équations (5) et (6) la variable z' est paramétrique.

Nous pouvons aussi chercher une fonction u_1' d'une équation analogue à (5), où A, B seront remplacées par A', B' et z sera remplacée par z' dans la dérivée qui est le facteur de C; l'équation (6) sera remplacée respectivement. Les fonctions u_1, u_2 ou u_1', u_2' sont nécessaires pour la construction de l'intégrale générale.

3. Pour l'équation à deux fonctions z, z' des trois arguments x_1, x_2, x_3:

$$A_1p_1 + A_2p_2 + A_3p_3 + C + A_1'p_1' + A_2'p_2' + A_3'p_3' + \\ + D_1(p_2p_3' - p_3p_2') + D_2(p_3p_1' - p_1p_3') + D_3(p_1p_2' - p_2p_1') = 0, \qquad (7)$$

l'équation (5) s'écrira:

$$A_1\frac{\partial f}{\partial x_1} + A_2\frac{\partial f}{\partial x_2} + A_3\frac{\partial f}{\partial x_3} - C\frac{\partial f}{\partial z} = 0, \qquad (8)$$

et au lieu de (6), suivant les notations:

$$u_{11} = \frac{\partial u_1}{\partial x_1}, \quad \ldots, \quad u_{14} = \frac{\partial u_1}{\partial z},$$

nous aurons:

$$(A_1'u_{14} - D_2u_{13} + D_3u_{12})\frac{\partial u}{\partial x_1} + (A_2'u_{14} - D_3u_{11} + D_1u_{13})\frac{\partial u}{\partial x_2} + \\ + (A_3'u_{14} - D_1u_{12} + D_2u_{11})\frac{\partial u}{\partial x_3} - (A_1'u_{11} + A_2'u_{12} + A_3'u_{13})\frac{\partial u}{\partial z} = 0. \qquad (9)$$

Les équations linéaires auxiliaires pour deux fonctions inconnues des plusieurs arguments $x_1, \ldots, x_m$ s'obtiendront d'une manière analogue. Par exemple, pour trois fonctions inconnues z, z', z'' des deux arguments x, y en prenant les relations:

$$u_1(x, y, z, z', z'') = C_1; \quad u_2(x, y, z, z', z'') = C_2; \quad n_3(x, y, z, z', z'') = C_3,$$

et en introduisant les jacobiens du troisième ordre, l'équation:

$$Ap + Bq + C + A'p' + B'q' + A''p'' + B''q'' + D_1(p'q'' - p''q') + \\ + D_2(p''q - pq'') + D_3(pq' - p'q) = 0,$$

s'écrira:

$$(xz'z'')A + (yz'z'')B + (zxz'')A' + (zyz'^1)B' + (zz'x)A'' + \\ + (zz'y)B'' - (zz'z'')C - (zxy)D_1 - (yz'x)D_2 - (xyz'')D_3 = 0.$$

Trois différentielles totales $du_1 = 0$, $du_2 = 0$, $du_3 = 0$ conduisent à la détermination d'une intégrale particulière u_1 de l'équation de la forme (5) avec les variables paramétriques z', z'' et à la recherche de l'intégrale particulière v_1 de l'équation linéaire par rapport aux jacobiens avec deux fonctions inconnues u_2, u_3 des arguments x, y, z, z', z''. Les équations à trois fonctions de 3, ..., m arguments etc. s'intègrent d'une manière analogue.

4. Deux méthodes sont connues pour l'intégration des systèmes d'équations linéaires: la méthode de Hamburger (1) avec mes modifications pour les équations homogènes, quand elle n'est pas applicable (2), et la mienne (3). Cette dernière est exposée pour les équations linéaires par rapport aux dérivées. Elle est applicable aussi pour les équations linéaires par rapport aux jacobiens: les coefficients des équations algébriques et différentielles auxiliaires seront seulement remplacés par des coefficients nouveaux, que l'on obtient sans peine. Les deux méthodes exigent l'intégration des systèmes d'équations linéaires auxiliaires à une seule fonction, tandis que dans la méthode exposée il faut intégrer chaque fois une seule équation linéaire à une fonction. Dans la méthode nouvelle nous

pouvons intégrer séparément chacune des n équations à n fonctions $z_1, \ldots, z_n$. Enfin, ici nous n'avons pas besoin de résoudre les équations algébriques du n-ème dégré pour n fonctions, comme dans les méthodes précédentes.

Cas particuliers: 1) quand les coefficients D sont nuls, nous avons une méthode d'intégration des équations linéaires par rapport aux dérivées; 2) quand

$$D = D_1 = D_2 = \ldots = A' = B' = A'_1 = A'_2 = \ldots = B'_1 = B'_2 = \ldots = 0,$$

nous avons la méthode connue de la théorie classique d'intégration des équations linéaires du premier ordre à une seule fonction inconnue z.

5. Exemple 1. Pour le système $xq - q' = 0$; $(xy + 1)p - yp' + yz = 0$ la méthode de Hamburger n'est pas applicable. Ma première méthode conduit à l'intégrale générale:

$$z = y\varphi(x) + \psi(y); \quad z' = (xy + 1)\varphi(x) + x\psi(y).$$

L'équation (5) donne $u_1 = z' - xz$, et de l'équation (6) nous avons $u_2 = x$ et nous obtenons $z' - xz = \varphi(x)$. La deuxième équation conduit aux intégrales $u_3 = (xy + 1)z - yz'$; $u_4 = y$ et donne $(xy + 1)z - yz' = \psi(y)$.

Exemple 2.

$$-x_1x_2p_2 + x_1x_3p_3 + x_2zp'_2 - x_3zp'_3 + x_1x_2(p_1p'_2 - p_2p'_1) + x_1x_3(p_3p'_1 - p_1p'_3) = 0.$$

L'équation (8) conduit à $u_1 = x_1 + z'$, et l'équation (9) conduit à $u_2 = x_2x_3$. L'équation (8) donne encore $u_2 = x_2x_3$; $n_3 = x_1z$; les deux intégrales vérifient (9). L'intégrale générale est:

$$x_1z = \varphi(x_1 + z'_1; x_2x_3).$$

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Manuscrit reçu
le 14. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHEMATIK

ZUR LAPLACE-TRANSFORMATION. IV

Von V. S. IGNATOVSKIJ (W. IGNATOWSKY), korrespondierendem Mitglied der Akademie

Als Fortsetzung unserer Note III (1) wollen wir nun den Aussenraum der Kugel mit Radius r_1 untersuchen. Deshalb haben wir jetzt statt der Bedingung (12) Note III:

$$r_1 \leqslant r \leqslant \infty \quad \text{und} \quad t > 0, \tag{1}$$

und statt der Grössen (13) kommen für uns nur die folgenden vier in Betracht:

$$q_n(r_1 + 0,\ t) = q_{n,1}(t); \quad \frac{\partial q_n}{\partial r}_{r \to r_1} = h_{n,1}(t), \tag{2}$$

$$q_n(r,\ 0) = g_n(r); \quad \frac{\partial q_n}{\partial t}_{t \to 0} = G_n(r). \tag{3}$$

Die Beziehung (15) bleibt bestehen, und es handelt sich nun um die Integration der Gleichung (16), wobei μ und ν durch (17) gegeben sind.

Von den obigen vier Grössen (2) und (3) können, wie wir weiter sehen werden, drei gegeben sein.

Wir multiplizieren nun (16) der Note III beiderseits mit r^{n+1} und führen die Funktion

$$r^{n+1} p_n = \pi_n(r,\ t) \tag{4}$$

ein, für welche wir dann die Gleichung:

$$a \frac{\partial^2 \pi_n}{\partial t^2} + \nu^2 \pi_n - \frac{\partial^2 \pi_n}{\partial r^2} + \frac{2n}{r} \frac{\partial \pi_n}{\partial r} = r^{n+1} \zeta_n(t,\ r), \quad a = \frac{1}{c^2} \tag{5}$$

erhalten. Hierauf wenden wir dann die Laplace-Transformation in bezug auf die Zeit t an:

$$\varphi_n(z,\ r) = \int_0^\infty e^{-zt} \pi_n(r,\ t)\, dt, \tag{6}$$

weshalb dan aus (5) fliesst

$$\frac{\partial^2 \varphi_n}{\partial r^2} - \frac{2n}{r} \frac{\partial \varphi_n}{\partial r} - a\left(z^2 + \frac{\nu^2}{a}\right)\varphi_n =$$

$$= -r^{n+1} \left\{ \nu_n(z, r) + azf_n(r) + aF_n(r) \right\}. \tag{7}$$

Hierbei bedeutet $\nu_n(z, r)$ die entsprechende Unterfunktion von $\zeta_n(t, r)$ und

$$f_n(r) = p_n(r, 0); \quad F_n(r) = \frac{\partial p_n}{\partial t}_{t \to 0}. \tag{8}$$

Ferner unterwerfen wir (7) einer weiteren Laplace-Transformation, jetzt aber in bezug auf r, und zwar wird wegen (1) sein:

$$\chi_n(z, \varepsilon) = \int_{r_1}^{\infty} e^{-\varepsilon r} \varphi_n(z, r)\, dr. \tag{9}$$

Aus (7) ergibt sich dann:

$$\frac{d\chi_n(z, \varepsilon)}{d\varepsilon} + \frac{2\varepsilon(1+n)}{\varepsilon^2 - k^2}\chi_n(z, \varepsilon) =$$

$$= \frac{e^{-\varepsilon r_1}}{\varepsilon^2 - k^2}\left\{(2n + 1 - \varepsilon r_1)\varphi_n(z, r_1) - r_1\varphi_n'(z, r_1)\right\} - \frac{L_n(z, \varepsilon)}{\varepsilon^2 - k^2}. \tag{10}$$

Hierbei bedeuten

$$k^2 = az^2 + \varrho^2; \quad \varphi_n'(z, r_1) = \frac{\partial \varphi_n(z, r)}{\partial r}_{r \to r_1} \tag{11}$$

und

$$L_n(z, \varepsilon) = \int_{r_1}^{\infty} e^{-\varepsilon r} M_n(z, r)\, dr \tag{12}$$

mit

$$M_n(z, r) = r^{n+2}\left\{\nu_n(z, r) + azf_n(r) + aF_n(r)\right\}. \tag{13}$$

Das allgemeine Integral von (10) lautet

$$\chi_n(z, \varepsilon) = \frac{C_n}{(\varepsilon^2 - k^2)^{n+1}} + \frac{(2n+1)\varphi_n(z, r_1) - r_1\varphi_n'(z, r_1)}{(\varepsilon^2 - k^2)^{n+1}} \cdot \int e^{-\varepsilon r_1}(\varepsilon^2 - k^2)^n\, d\varepsilon -$$

$$- \frac{r_1\varphi_n(z, r_1)}{(\varepsilon^2 - k^2)^{n+1}} \int e^{-\varepsilon r_1}\varepsilon(\varepsilon^2 - k^2)^n\, d\varepsilon + \frac{1}{(\varepsilon^2 - k^2)^{n+1}} \int L_n(z, \varepsilon)(\varepsilon^2 - k^2)^n\, d\varepsilon. \tag{14}$$

Da die Integrationskonstante C_n natürlich nicht von ε abhängt, so ist sie auch unabhängig von r. Dieses Integral lässt sich durch endliche Reihen darstellen, aber die weiteren Rechnungen werden bei beliebigem n sehr kompliziert und wir wollen uns vorläufig auf die Fälle $n = 0$ und $n = 1$ beschränken.

Hierfür gehen wir mit (14) in den Oberraum in bezug auf r zurück und erhalten dann, erstens:

$$C_0 = C_1 = 0, \tag{15}$$

und ferner:

$$\varphi_0(z, r) = \varphi_0(z, r_1)\operatorname{ch} k(r - r_1) + \varphi_0'(z, r_1)\frac{\operatorname{sh} k(r - r_1)}{k} -$$

$$- \int_{r_1}^{r} \frac{\operatorname{sh} k(r - \tau)}{k} \frac{M_0(z, \tau)\, d\tau}{\tau} \tag{16}$$

und

$$\varphi_1(z, r) = \frac{\varphi_1(z, r_1)}{r_1}\left\{r \operatorname{ch} k(r - r_1) - \frac{\operatorname{sh} k(r - r_1)}{k}\right\} +$$

$$+\frac{\varphi_1'(z,\,r_1)}{r_1 k}\left\{(r-r_1)\operatorname{ch} k(r-r_1)-\frac{\operatorname{sh} k(r-r_1)}{r_1 k^2}+r\operatorname{sh} k(r-r_1)\right\}-$$

$$-\int_{r_1}^{r}\frac{M_1(z,\,\tau)}{k^3\tau^3}\left\{k(r-\tau)\operatorname{ch} k(r-\tau)+(k^2\tau r-1)\operatorname{sh} k(r-\tau)\right\}d\tau, \qquad (17)$$

und dies sind also die entsprechenden Integrale von (7).

In (16) und (17) kommen alle vier Grössen $\varphi_n(z, r_1)$, $\varphi_n'(z, r_1)$, $f_n(r)$ und $F_n(r)$ $(n = 1, 2)$ vor, entsprechend (2) und (3). Für $r = r_1$ erhalten wir Identitäten und ebenfalls nach Differentiation nach r und bei $r = r_1$.

Wir können deshalb keine von diesen vier Grössen eliminieren und müssen zu diesem Zweck anders verfahren. Hierzu denken wir uns den Übergang zum Unterraum in (5) in bezug auf r ausgeführt d. h. laut

$$\gamma_n(t,\,\varepsilon)=\int_{r_1}^{\infty}e^{-\varepsilon r}\pi_n(r,\,t)\,dr. \qquad (18)$$

Gehen wir anderseits in (14) zu dem Oberraum in bezug auf t zurück, so müssen wir augenscheinlich ebenfalls $\gamma_n(t, \varepsilon)$ erhalten. Tun wir dies für $n = 0$ $(C_0 = 0)$, so erhalten wir:

$$\gamma_0(t,\,\varepsilon)=-\frac{e^{-\varepsilon r_1}}{ak_1}\int_0^t\operatorname{sh}(k_1\tau)\cdot\pi_0'(r_1,\,t-\tau)\,d\tau-$$

$$-\frac{e^{-r_1\varepsilon}}{ak_1^2}\frac{\partial}{\partial t}\int_0^t\operatorname{ch}(k_1\tau)\cdot\pi_0(r_1,\,t-\tau)\,d\tau+\frac{\varepsilon e^{-r_1\varepsilon}}{ak_1^2}\pi_0(r_1\,t)+$$

$$+\frac{1}{ak_1}\int_0^t\operatorname{sh}(k_1\tau)\cdot\left\{\int_{r_1}^{\infty}e^{-\varepsilon r}\zeta_0(t-\tau,\,r)dr\right\}d\tau+\frac{\partial}{\partial t}\left\{\frac{\operatorname{sh}(k_1 t)}{k_1}\int_{r_1}^{\infty}xe^{-\varepsilon x}f_0(x)dx\right\}+$$

$$+\frac{\operatorname{sh}(k_1 t)}{k_1}\int_{r_1}^{\infty}xe^{-\varepsilon x}F_0(x)\,dx, \qquad (19)$$

wobei

$$k_1=c\sqrt{\varepsilon^2-\nu^2};\quad \pi_0'(r_1,\,t)=\frac{\partial\pi(r,\,t)}{\partial r}_{r\to r_1}. \qquad (20)$$

Wir gehen nun in (19) zu dem Oberraum in bezug auf r zurück und setzen hinterher $r = r_1$. Wir erhalten demnach $\pi_0(r_1, t)$. Es ergibt sich dann, erstens für $t = 0$

$$\pi_0(r_1,\,0)-r_1 f_0(r_1)=0;\quad \text{resp. } q_{0,1}(0)-g_0(r_1)=0, \qquad (21)$$

und dann, falls wir wieder zu dem Unterraum in bezug auf t übergehen:

$$\varphi_0(z,\,r_1)=-\frac{\varphi_0'(z,\,r_1)}{k}+2\left\{\beta_1(z)+\beta_2(z)+\beta_3(z)\right\}, \qquad (22)$$

wobei die Ausdrücke $\beta_n(z)$, $(n = 1, 2, 3)$ von $\zeta_0(r, t)$, $f_0(r)$ und $F_0(r)$ abhängen und mit denselben verschwinden.

Aus (22) und (16) können wir nun z. B. $\varphi_0'(z, r_1)$ eliminieren und erhalten

$$\varphi_0(z,\,r)=\varphi_0(z,\,r_1)\,e^{-k(r-r_1)}+2\operatorname{sh} k(r-r_1)\cdot\left\{\beta_1(z)+\beta_2(z)+\beta_3(z)\right\}-$$

$$-\int_{r_1}^{r}\frac{\operatorname{sh} k(r-\tau)}{k}\cdot\frac{M_0(z,\,\tau)}{\tau}\,d\tau. \qquad (23)$$

Hier müssen wir nun zu dem Oberraum in bezug auf t übergehen, um den gesuchten Wert von $\pi_0(r, t)$ zu erhalten, der jetzt nur von den drei Grössen $\pi_0(r_1\, t)$, $f_0(r)$ und $F_0(r)$ abhängen wird, also von drei der Grössen (2) und (3), wie anfangs behauptet. Ferner müssen wir jetzt von π_0 zu q_0 übergehen und erhalten dann endgültig, falls wir die Bezeichnungen:

$$M(x, y) = I_0(\varrho\sqrt{x^2 - c^2y^2}); \qquad r - r_1 = x, \tag{24}$$

einführen,

$$rq_0(r, t) = \frac{e^{-\mu t}}{2}\Big\{(x + ct + r_1)\, g_0(x + ct + r_1) - (ct - x + r_1)\, g_0(ct - x + r_1)\Big\} +$$

$$+ r_1 e^{\frac{-x\mu}{c}} q_{0,1}\left(t - \frac{x}{c}\right) - cr_1 \int_{\frac{x}{c}}^{t} e^{-y\mu} q_{0,1}(t - y)\, \frac{\partial M(x, y)}{\partial x}\, dy +$$

$$+ \frac{e^{-\mu t}}{2c} \int_{-ct}^{x} (x - y + r_1)\Big[\Big\{ G_0(x - y + r_1) +$$

$$+ \mu g_0(x - y + r_1)\Big\} M(y, t) + g_0(x - y + r_1)\, \frac{\partial M(y, t)}{\partial t}\Big]\, dy -$$

$$- \frac{e^{-\mu t}}{2c} \int_{x}^{ct} (y - x + r_1)\Big[\Big\{ G_0(y - x + r_1) + \mu g_0(y - x + r_1)\Big\} M(y, t) +$$

$$+ g_0(y - x + r_1)\, \frac{\partial M(y, t)}{\partial t}\Big]\, dy + \frac{c}{2} \int_{0}^{\frac{x}{c}} e^{-\mu y}\, dy \int_{-cy}^{cy} S(y, \tau)\, d\tau +$$

$$+ \frac{c}{2} \int_{\frac{x}{c}}^{t} e^{-\mu y}\, dy \int_{-cy}^{x} S(y, \tau)\, d\tau + \frac{c}{2} \int_{\frac{x}{c}}^{t} e^{-\mu y}\, dy \int_{cy}^{x} S(y, \tau)\, d\tau; \quad \underline{ct > x.} \tag{25}$$

mit

$$S(y, \tau) = (x - \tau + r_1)\, \eta_0(t - y,\ x - \tau + r_1)\, M(\tau, y), \tag{26}$$

und

$$rq_0(r, t) = \frac{e^{-\mu t}}{2}\Big\{(x - ct + r_1)\, g_0(x - ct + r_1) + (x + ct + r_1)\, g_0(x + ct + r_1)\Big\} +$$

$$+ \frac{e^{-\mu t}}{2c} \int_{-ct}^{ct} (x - y + r_1)\Big[\Big\{ G_0(x - y + r_1) + \mu g_0(x - y + r_1)\Big\} M(y, t) +$$

$$+ g_0(x - y + r_1)\, \frac{\partial M(y, t)}{\partial t}\Big]\, dy + \frac{c}{2} \int_{0}^{t} e^{-\mu y}\, dy \int_{-cy}^{cy} S(y, \tau)\, d\tau; \quad \underline{x > tc.} \tag{27}$$

Der stetige Übergang von (25) zu (27) bei $x = ct$ ergibt sich auf Grund von (21).

Wir gehen jetzt zu dem Fall $n = 1$ über und setzen zur Vereinfachung $g_1(r) = G_1(r) = \zeta_1(t, r) = 0$, d. h. es ist nur $q_{1,1}(t)$ gegeben. Es folgt:

$$r^2 q_1(r, t) = rr_1 e^{-\frac{x\mu}{c}} q_{1,1}\left(t - \frac{x}{c}\right) - crr_1 \int_{\frac{x}{c}}^{t} e^{-y\mu}\, q_{1,1}(t - y)\, \frac{\partial M(x, y)}{\partial x}\, dy -$$

$$- \frac{c^3 x}{b}\, \frac{\partial^2 Q}{\partial x^2} - \frac{c^3 x}{r_1 b}\, \frac{\partial Q}{\partial x}; \qquad \underline{ct > x,} \tag{28}$$

und gleich Null für $ct < x$. Hierbei bedeuten

$$Q = \int_{\frac{x}{c}}^{t} q_{1,1}(t-y)\, e^{-y\mu}\, dy \int_{\frac{x}{c}}^{y} M(x, \tau) \operatorname{sh} b\,(y-\tau)\, d\tau, \tag{29}$$

und

$$b = \frac{c}{r_1}\sqrt{1-\rho^2 r_1^2}. \tag{30}$$

Machen wir in (25) und (27) $g_0 = G_0 = \eta_0 = 0$; $\mu = 0$ und setzen $q_{0,1} = \text{const} = C$ so erhalten wir:

und
$$q_0(r, t) = \begin{cases} \frac{Cr_1}{r}; & ct > x. \\ 0; & ct < x. \end{cases} \tag{31}$$

und dies genügt der Laplaceschen Gleichung $\Delta u = 0$ $(ct \neq x)$. Wir haben also hier den Übergang zu dem statischen Fall.

Auch in der Note I [2] haben wir von den dortigen vier Grössen (2) und (3) eine eliminiert. Der Unterschied mit der vorliegenden Arbeit besteht darin, dass wir hier diese Elimination im Unterraum ausgeführt haben, was bedeutend einfacher ist, als im Oberraum, wie in der Note I.

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Eingegangen
d. 20. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHEMATICS

ON SOME DEVELOPMENTS INTO FOURIER-BESSEL SERIES

By N. S. KOŠLJAKOV (KOSHLIAKOV), Corresponding Member of the Academy

In his work entitled «The Use of Series of Bessel Functions in Problems Connected with Cylindrical Wind-Tunnels» (1) G. N. Watson gave formulae transforming the series:

$$\left.\begin{aligned} S_1 &= \sum_{m=1}^{\infty} \frac{J_0(k_m r)}{J_1^2(k_m a)} e^{-k_m x}, \\ S_2 &= \sum_{m=1}^{\infty} \frac{J_1(k_m r)}{k_m a J_1^2(k_m a)} e^{-k_m x}, \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} S_3 &= \sum_{m=1}^{\infty} \frac{J_0(n_m r)}{J_0^2(n_m a)} e^{-n_m x}, \\ S_4 &= \sum_{m=1}^{\infty} \frac{J_0'(n_m r)}{n_m a J_0^2(n_m a)} e^{-n_m x}, \end{aligned}\right\} \quad (2)$$

into new series arranged according to the powers of x and facilitating the calculation of the sums S_1, S_2, S_3, S_4 at small values of x. In formulae (1) k_1, k_2, ... denote the positive roots of the equation $J_0(ka) = 0$, while in (2) n_1, n_2, ... denote the positive roots of the equation $J_1(na) = 0$.

The transformations given by Watson are a special case of the general summation formula established by me in 1920 in my work entitled «On Some Applications of the Theory of Integral Residues» (2). In the present paper I shall point out a new form of this summation formula and discuss some cases of its application.

Let λ_1, λ_2, ... denote the positive roots of the equation:

$$J_n(\lambda) = 0. \quad (3)$$

Then, under certain conditions imposed on function $\varphi(z)$* the following summation formula holds:

* These conditions are stated on p. 28—30 of my above mentioned work. They are not reproduced here as it was intended by us to employ a modification of Formula (5) and to state the conditions imposed upon the function $f(z)$.

$$\sum_{j=1}^{\infty} \frac{\lambda_j^{n-1}}{J_{n+1}(\lambda_j)} \varphi(\lambda_j) = 2^{n-1}\Gamma(n+1)\varphi(0) + \frac{1}{\pi}\int_0^{\infty} \frac{\varphi(ix)-\varphi(-ix)}{2i} \frac{x^{n-1}}{I_n(r)}\,dx, \quad (4)$$

where

$$I_n(x) = e^{-\frac{n\pi i}{2}} J_n\left(xe^{\frac{\pi i}{2}}\right).$$

Let us replace in formula (4) the function $\varphi(z)$ by the function:

$$\frac{\pi}{2} z^{p-n+2} \left\{J_n(z)\, Y_p(\alpha z) - Y_n(z)\, J_p(\alpha z)\right\} f(z),$$

where n and p are integers, $p \geqslant n-1$, α is a real number $(0 < \alpha < 1)$, while the function $f(z)$ of the complex variable $z = x + iy$ satisfies the following conditions:

1) $f(x+iy)$ is holomorphous at any $x \geqslant 0$;

2) the equality

$$\lim_{R\to\infty} \left\{R^{p-n+\frac{5}{2}} e^{-\alpha R|\sin\theta|} |f(Re^{\theta i})|\right\} = 0, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

is satisfied.

Then, formula (4) is transformed into the following formula:

$$\sum_{j=1}^{\infty} \lambda_j^p \frac{J_p(\alpha\lambda_j)}{J_{n+1}^2(\lambda_j)} f(\lambda_j) = \varepsilon 2^{n-1}\Gamma(n+1)\alpha^{n-1} f(0) +$$

$$+ \frac{(-1)^n}{\pi}\int_0^{\infty} x^{p+1} \frac{(-1)^p I_p(\alpha x) K_n(x) - (-1)^n K_p(\alpha x) I_n(x)}{I_n(x)} \frac{f(ix)-f(-ix)}{2i}\,dx, \quad (5)$$

where $K_n(x)$ denotes Macdonald's function, while $\varepsilon = 0$ when $p > n$ and $\varepsilon = 1$ when $p = n-1$.

Putting in formula (5) $f(z) = z^{m+\frac{1}{2}} K_{m+\frac{1}{2}}(\beta z)$, where m is a positive integer or zero, we get

$$\sum_{j=1}^{\infty} \lambda_j^{p+m+\frac{1}{2}} \frac{J_p(\alpha\lambda_j)\, K_{m+\frac{1}{2}}(\beta\lambda_j)}{J_{n+1}^2(\lambda_j)} = \frac{\varepsilon 2^{n+m-\frac{3}{2}}\Gamma(n+1)\Gamma\left(m+\frac{1}{2}\right)\alpha^{n-1}}{\beta^{m+\frac{1}{2}}} +$$

$$+ \frac{2^{p+m-\frac{1}{2}}\Gamma\left(p+m+\frac{3}{2}\right)\alpha^p\beta^{m+\frac{1}{2}}}{(\alpha^2+\beta^2)^{p+m+\frac{3}{2}}} +$$

$$+ \frac{(-1)^{p-n+1}}{2}\int_0^{\infty} \frac{K_n(x)}{I_n(x)} I_p(\alpha x)\, J_{m+\frac{1}{2}}(\beta x)\, x^{p+m+\frac{3}{2}}\,dx. \quad (6)$$

If we develop $I_p(\alpha x)\, J_{m+\frac{1}{2}}(\beta x)$ into a power series, we get the following transformation*

* The proof of this transformation for the case $m = 0$ was the subject of the thesis of L. S. Srol, student of the Leningrad University, written under my guidance.

$$\sum_{j=1}^{\infty} \lambda_j^{p+m+\frac{1}{2}} \frac{J_p(\alpha\lambda_j)\, K_{m+\frac{1}{2}}(\beta\lambda_j)}{J_{n+1}^2(\lambda_j)} = \frac{\varepsilon 2^{n+m-\frac{3}{2}} \Gamma(n+1)\, \Gamma\left(m+\frac{1}{2}\right) \alpha^{n-1}}{\beta^{m+\frac{1}{2}}} +$$

$$+ \frac{2^{p+m-\frac{1}{2}} \Gamma\left(p+m+\frac{3}{2}\right) \alpha^p \beta^{m+\frac{1}{2}}}{(\alpha^2+\beta^2)^{p+m+\frac{3}{2}}} +$$

$$+ (-1)^{p-n+1} \frac{\pi}{2} \left(\frac{\alpha}{2}\right)^p \left(\frac{\beta}{2}\right)^{m+\frac{1}{2}} \sum_{k=1}^{\infty} (-1)^{k-1} \lambda_{2(p+k+m)} \times$$

$$\times \sum_{s=0}^{k=1} \frac{(-1)^s \alpha^{2s} \beta^{2(k-s-1)}}{s!\, \Gamma(p+s+1)\, \Gamma(k-s)\, \Gamma\left(m+k-s+\frac{1}{2}\right)}, \qquad (7)$$

where for brevity we put

$$\lambda_{2r} = \frac{1}{\pi(2r+1)} \int_0^{\infty} \frac{x^{2r} dx}{J_n^2(x)}. \qquad (8)$$

If now we put:

$$m=0, \quad n=0, \quad p=0;$$
$$m=0, \quad n=0, \quad p=-1;$$
$$m=0, \quad n=1, \quad p=0,$$

then formula (7) yields the transformation pointed out by Watson for the sums S_1, S_2 and S_3. A slight modification of the first member of the right-hand side of equation (5) gives Watson's result for the sum S_4 (when $m=0$, $n=1$, $p=-1$).

The summation formulae (4) and (5) enable us to study the analytical nature of the functions given by the series of Bessel functions.

Thus, for instance, if we put:

$$f(z) = \frac{1}{z^{s+p}}, \quad R(s) > \frac{1}{2},$$

and modify formula (5) so that the integration should proceed along a straight line parallel to the imaginary axis and intersecting the real axis at point τ, where $0 < \tau < 1$, then, introducing for brevety the notation:

$$\zeta_\alpha(s) = \sum_{j=1}^{\infty} \frac{J_p(\alpha\lambda_j)}{J_{n+1}^2(\lambda_j)} \frac{1}{\lambda_j^s}, \qquad (9)$$

we find the expression:

$$\zeta_\alpha(s) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{\pi}{2} \frac{J_n(z)\, Y_p(\alpha z) - Y_n(z)\, J_p(\alpha z)}{J_n(z)} z^{1-s} dz, \qquad (10)$$

which gives an analytical representation of the function $\zeta_\alpha(s)$ over the whole plane of the complex variable s.

If we assume that

$$R(s) < -p \text{ if } p \geqslant n \text{ and } R(s) < -p-1 \text{ if } p = n-1,$$

we may, in formula (10), put τ equal to 0, as a result of which we obtain the following formula:

$$\zeta_\alpha(1-p-\sigma) = \frac{(-1)^n}{\pi}\cos\frac{\pi\sigma}{2}\int_0^\infty x^{p+1}\,\frac{(-1)^p I_p(\alpha x)K_n(x)-(-1)^n K_p(\alpha x)I_n(x)}{I_n(x)}\,x^{\sigma-1}\,dx, \quad (11)$$

it being possible to prove by means of analytical continuation that this formula holds good when

$$R(\sigma)>0 \text{ if } p\geqslant n \text{ and } R(\sigma)>1 \text{ if } p=n-1.$$

Formula (11) shows that

$$\zeta_\alpha(-2k-p)=0, \quad (12)$$

if $p\geqslant n-1$ and $k=1, 2, 3, \ldots$

If $p>n$, formula (12) holds also in the case of $k=0$. If, on the other hand, $p=n-1$, then $\zeta_\alpha(-p)\neq 0$, and in this case

$$\zeta_\alpha(-n+1)=2^{n-1}\Gamma(n+1)\alpha^{n-1}. \quad (13)$$

Further, from formula (11) it follows that

$$\zeta_\alpha[-(2k-1+p)] = (-1)^{k-1}\frac{2^{2k+p-1}\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+p+\frac{1}{2}\right)}{\pi\cdot\alpha^{2k+p+1}} + \\ +(-1)^{p+n+k}\sum_{s=0}^{\infty}\frac{\lambda_{2(p+s+k)}}{s!\,\Gamma(p+s+1)}\left(\frac{\alpha}{2}\right)^{p+2s}. \quad (14)$$

The properties of the function $\zeta_\alpha(s)$ may be made use of in deducing summation formulae analogous to formulae (4) and (5). Thus, for instance, employing the Riemann's-Mellin's inversion formula, we find that

$$\sum_{j=1}^{\infty}\lambda_j^{p+m+\frac{1}{2}}\frac{J_p(\alpha\lambda_j)\,K_{m+\frac{1}{2}}(\beta\lambda_j)}{J_{n+1}^2(\lambda_j)} = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\Gamma\left(\frac{p+s}{2}\right)\Gamma\left(\frac{p+2m+s+1}{2}\right)}{2^{\frac{3}{2}-s-p-m}\,\beta^{p+s+m+\frac{1}{2}}}\zeta_\alpha(s)\,ds,$$

where $\tau>\frac{1}{2}$.

By describing from point τ a semi-circumference intersecting the real axis in its negative portion and increasing the radius of this semi-circumference to infinity, we find that

$$\sum_{j=1}^{\infty}\lambda_j^{p+m+\frac{1}{2}}\frac{J_p(\alpha\lambda_j)\,K_{m+\frac{1}{2}}(\beta\lambda_j)}{J_{n+1}^2(\lambda_j)} = \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{2}\,\beta^{m+\frac{1}{2}}}\zeta_\alpha(-p)+ \\ +(-1)^{m-1}\sqrt{\frac{\pi}{2}}\sum_{k=1}^{\infty}\frac{\Gamma(m+k)\,\zeta_\alpha[-(2m+2k+p-1)]}{\Gamma(k)\,\Gamma(2m+2k)}\,\beta^{2k+m-\frac{3}{2}}. \quad (15)$$

Taking into consideration formula (14), we come again to formula (7).

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Received
29. XI. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHEMATIK

ÜBER DIE NUMERISCHE LÖSUNG DER DIFFERENTIALGLEICHUNG

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\varphi(x, y, z)$$

Von Sch. MIKELADZE

(Vorgelegt von A. Krylov; Mitglied der Akademie, d. 16. XII. 1936)

Es werde die Funktion $u(x, y, z)$ gesucht, die im von der geschlossenen Fläche S begrenzten Gebiete D der Differentialgleichung:

$$\Delta u=\varphi(x, y, z) \qquad \left(\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right),$$

und auf der Fläche S der Bedingung $u=f(x, y, z)$ genügt.

Zur numerischen Auflösung der gegebenen Gleichung konstruiren wir ein kubisches Gitter. Die Kanten der Gitterwürfel seien parallel zu den Koordinatenaxen.

Konstruiren wir den Flächen der Gitterwürfel folgend, eine geschlossene, am besten der Fläche S sich anschmiegende Fläche S'.

Betrachten wir das Würfelchen mit dem Mittelpunkt im Punkte x_0, y_0, z_0. Durch den Mittelpunkt des Würfelchens führen wir eine Schnittebene senkrecht zur Axe OZ. Der Durchschnitt ist ein Quadrat mit dem Mittelpunkt (x_0, y_0, z_0).

Das Quadrat des Würfelchens mit dem Mittelpunkt in (x_0, y_0, z_0+h) bezeichnen wir als das erste Quadrat, mit dem Mittelpunkt (x_0, y_0, z_0) als das zweite, und mit dem Mittelpunkt (x_0, y_0, z_0-h) als das dritte Quadrat. Die Seitenlänge der erwähnten Quadrate ist $2h$.

Es bezeichne u_k den u-Wert im Gitterknoten mit den Koordinaten x_k, y_k, z_k.

In diesem Aufsatze werden folgende neue Formel gegeben:

$$u_0=\frac{2\sum_i u_i+\sum_k u_k}{24}-\frac{h^2}{4}\Delta u_0-\frac{h^4}{48}\Delta^2 u_0, \tag{1}$$

$$u_0=\frac{8\sum_i u_i+\sum_r u_r}{56}-\frac{3h^2}{14}\Delta u_0-\frac{h^4}{56}\Delta^2 u_0, \tag{2}$$

$$u_0 = \frac{14\sum_i u_i + 3\sum_k u_k + \sum_r u_r}{128} - \frac{h^2}{128}\left[36\,\Delta u_0 - \sum_i \Delta u_i\right] - \frac{7h^4}{256}\Delta^2 u_0 - \frac{h^6}{768}\Delta^3 u_0, \qquad (3)$$

wo $u_0 = u(x_0, y_0, z_0)$; $\sum_i u_i$ bezeichnet die Summe aller möglichen u-Werte in den Seitenmitten des zweiten Quadrates und in den Mittelpunkten des ersten und dritten Quadrates, $\sum_k u_k$ — die Summe aller möglichen u-Werte in den Seitenmitten des ersten und dritten Quadrates und an den Ecken des zweiten Quadrates, $\sum_r u_r$ — die Summe aller möglichen u-Werte an den Ecken des ersten und dritten Quadrates, Δu_0 — den Wert von Δu im Punkte (x_0, y_0, z_0), $\sum_i \Delta u_i$ — die Summe aller möglichen Δu-Werte in den Seitenmitten des zweiten Quadrates und in den Mittelpunkten des ersten und dritten Quadrates; $\Delta^s u_0$ bezeichnet den symbolischen Ausdruck:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{(s)} u(x, y, z) \qquad (s = 2, 3),$$

im Punkte (x_0, y_0, z_0).

Die Restglieder der ersten zwei Formeln sind von 6-ter Kleinheitsordnung bezüglich h, der letzten Formel—von 8-ter Ordnung.

Mit Hilfe der Taylor'schen Formel finden wir:

$$\sum_i u_i = 6u_0 + h^2\Delta u_0 + \frac{h^4}{12}\left[\frac{\partial^4 u}{\partial x_0^4} + \frac{\partial^4 u}{\partial y_0^4} + \frac{\partial^4 u}{\partial z_0^4}\right] + 0(h^6). \qquad (4)$$

Legen wir durch den Punkt (x_0, y_0, z_0) Ebene parallel zu den Koordinatenebenen. Schreiben wir für die beim Durchschnitt erhaltenen Quadrate Beziehungen analog der Beziehung (13) in unserer Monographie(1).

Multiplizieren wir beide Seiten dieser Beziehungen und der Beziehung (4) mit den Faktoren α, β, γ, δ und summieren die Produkte. Durch passende Auswahl von α, β, γ, δ erhält man die Formel (1).

Gehen wir nun zur Ableitung der Formel (2) über. Schreiben wir die Beziehungen zwischen den u-Werten an den Ecken des ersten und dritten Quadrates analog der Beziehung (10) des zweiten Kapitels der Monographie (1), unter Beibehaltung der Glieder bis zur 6-ten Kleinheitsordnung (inclusive) bezüglich h. Nach einfacher Rechnung erhält man:

$$\sum_r u_r = 8u_0 + 4h^2\Delta u_0 + \frac{h^4}{3}\left[\frac{\partial^4 u}{\partial x_0^4} + \frac{\partial^4 u}{\partial y_0^4} + \frac{\partial^4 u}{\partial z_0^4} + 6\left(\frac{\partial^4 u}{\partial x_0^2 \partial y_0^2} + \frac{\partial^4 u}{\partial y^2 \partial z_0^2} + \frac{\partial^4 u}{\partial x_0^2 \partial z_0^2}\right)\right] + 0(h^6). \qquad (5)$$

Durch Multiplikation beider Teile (4) mit 8 und Addition zu (5) erhält man die Formel (2).

Leiten wir nun die Formel (3) ab. Eine Berechnung zeigt, dass

$$2h^2\sum_i \Delta u_i = 12h^2\Delta u_0 + 2h^4\Delta^2 u_0 + \frac{h^6}{6}\left[\frac{\partial^6 u}{\partial x_0^6} + \frac{\partial^6 u}{\partial y_0^6} + \frac{\partial^6 u}{\partial z_0^6} + \frac{\partial^6 u}{\partial x_0^4 \partial y_0^2} + \frac{\partial^6 u}{\partial y_0^4 \partial x_0^2} + \frac{\partial^6 u}{\partial y_0^4 \partial z_0^2} + \frac{\partial^6 u}{\partial z_0^4 \partial y_0^2} + \frac{\partial^6 u}{\partial x_0^4 \partial z_0^2} + \frac{\partial^6 u}{\partial z_0^4 \partial x_0^2}\right] + 0(h^8). \qquad (6)$$

Schreiben wir die Beziehungen (1) und (2) unter Beibehaltung von Glieder bis zur 8-ten Kleinheitsordnung (inclusive) bezüglich h. Multiplizieren wir diese Beziehungen wie auch die Beziehung (6) mit den Faktoren α, β, γ und summieren die Produkte. Durch entsprechende Auswahl von α, β, γ wird die Formel (3) erhalten.

Beziffern wir die inneren Punkte des Gitters in willkürlicher Folge von 1 bis n. Die Anzahl der Gleichungen von der Art (1), (2), (3) ist dann gleich der Anzahl der inneren Knoten des Gitters. Somit wird tatsächlich die Lösung der gegebenen Differentialgleichungen durch Berechnung von n algebraischen Lineargleichungen mit n Unbekannten ausgeführt.

Betrachten wir das Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

das die Fläche S umfasst [so dass (S) vollständig innerhalb des Ellipsoids liegt] und sich möglichst eng an die Fläche S anschmiegt. Seien die Halbaxen des Ellipsoids parallel zu den Koordinatenaxen gerichtet.

Unter Verallgemeinerung des Hilfsatzes § 4 der Monographie [1] und nach Ausführung der Rechnung finden wir, z. B., dass der Formel (2) folgende Abschätzung entspricht:

$$|\xi|\leqslant\eta+\frac{7}{80}\frac{M_6h^4}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},$$

wo h—die Grösse der Kante des Gitterwürfelchens ist, ξ—Fehler des Auflösungsgitters, η—genaue obere Grenze der Absolutwerte des Fehlers von u in den Grenzknoten des Gitters, und M_6 der Maximalwert der absoluten Werte der partiellen Ableitungen der 6-ten Ordnung von u im Gebiete D ist.

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Eingegangen
d. 16. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MATHEMATIK

ÜBER NUMERISCHE INTEGRATION DER LAPLACESCHEN UND POISSONSCHEN GLEICHUNGEN

Von SCH. MIKELADZE

(Vorgelegt von A. N. Krylov, Mitglied der Akademie, d. 28. XII. 1936)

Betrachten wir ein vom Umriss γ begrenztes Gebiet. Sei dieses Gebiet, für das die Dirichletsche Aufgabe gelöst wird, von einem Gitter bedeckt. Auf dem Gitter merken wir den längs den Gitterlinien sich hinziehenden, geschlossenen, gebrochenen Linienzug γ', der möglichst eng dem Umrisse γ folgt. Aus den verschiedenen Gitterarten wird man die Benutzung des Quadratgitters bevorzugen, da man mit Hilfe des letzteren am einfachsten die Konstruktion des den Umriss γ approximierenden Umrisses γ' durchführt.

Wir bemerken, dass, falls zur Lösung der Gleichung:

$$\Delta u = \varphi(x, y,) \qquad \left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

ein quadratisches Gitter unter Anwendung der C. Rungeschen (1) Beziehung benutzt wird,—die Genauigkeitsordnung des Auflösungsgitters gleich h^2 (h—Gitterlänge) ist, dagegen ist sie bei einem dreieckigen Gitter und der Gerschgorinschen (2) Beziehung oder bei einem quadratischen Gitter und der von uns aufgestellten Beziehung (3) gleich h^4.

In diesem Aufsatze werden neue Beziehungen zwischen u-Werten an den Knotenpunkten des quadratischen Gitters gegeben und bewiesen, dass diese Beziehungen die Werte von u mit einer Genauigkeit von der Ordnung h^6 ergeben.

Es sei u eine Funktion der Veränderlichen x und y. Es bezeichne u_k den u-Wert im Gitterpunkte (x_k, y_k). Wir haben:

$$u_0 = \frac{4[u_1 + u_2 + u_3 + u_4] + [u_5 + u_6 + u_7 + u_8]}{20} - \\ - \frac{h^2}{100}\left[34\Delta u_0 - \sum_i \Delta u_i\right] - \frac{7h^4}{200}\Delta^2 u_0 - \frac{h^6}{600}\Delta^3 u_0, \tag{1}$$

$$u_0 = \frac{4[u_1 + u_2 + u_3 + u_4] + [u_5 + u_6 + u_7 + u_8]}{20} - \\ - \frac{h^2}{400}\left[116\Delta u_0 + \sum_k \Delta u_k\right] - \frac{h^4}{50}\Delta^2 u_0 - \frac{h^6}{2\,400}\Delta^3 u_0, \tag{2}$$

wo $u_0 = u(x_0, y_0)$, u_1, u_2, u_3, u_4 die Werte von $u(x, y)$ in Punkten, die um h längs der Horizontalen und der Vertikalen von (x_0, y_0) abstehen, u_5, u_6, u_7, u_8 die $u(x, y)$-Werte an den Ecken des Quadrates mit dem Mittelpunkt im Punkte (x_0, y_0) und einer Seitendimension $2h$ bedeuten; $\sum_i \Delta u_i$ und $\sum_k \Delta u_k$ bezeichnen entsprechend die Summe aller möglichen

Werte von Δu in den Seitenmitten und an den Ecken des betrachteten Quadrats, und $\Delta^S u_0$, endlich, bezeichnet abgekürzt den Wert des symbolischen Ausdruckes:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^{(s)} u(x, y) \qquad (s=2, 3)$$

im Punkte (x_0, y_0).

Vorausgesetzt ist, dass die Seiten des Quadrats parallel zu den Koordinatenaxen gerichtet sind. Die Restglieder der Formeln (1) und (2) sind von 8-ter Kleinheitsordnung bezüglich h.

In der Tat, die Berechnung zeigt, dass

$$4[u_1+u_2+u_3+u_4]+[u_5+u_6+u_7+u_8]=20u_0+6h^2\Delta u_0+$$
$$+\frac{h^4}{2}\Delta^2 u_0+\frac{h^6}{60}\left[\frac{\partial^6 u}{\partial x_0^6}+5\frac{\partial^6 u}{\partial x_0^4\partial y_0^2}+5\frac{\partial^6 u}{\partial x_0^2\partial y_0^4}+\frac{\partial^6 u}{\partial y_0^6}\right]+O(h^8), \qquad (3)$$

$$h^2\sum_i \Delta u_i=4h^2\Delta u_0+h^4\Delta^2 u_0+\frac{h^6}{12}\left[\frac{\partial^6 u}{\partial x_0^6}+\frac{\partial^6 u}{\partial x_0^4\partial y_0^2}+\frac{\partial^6 u}{\partial x_0^2\partial y_0^4}+\frac{\partial^6 u}{\partial y_0^6}\right]+O(h^8), \qquad (4)$$

$$h^2\sum_k \Delta u_k=4h^2\Delta u_0+2h^4\Delta^2 u_0+$$
$$+\frac{h^6}{6}\left[\frac{\partial^6 u}{\partial x_0^6}+7\frac{\partial^6 u}{\partial x_0^4\partial y_0^2}+7\frac{\partial^6 u}{\partial x_0\partial y_0^4}+\frac{\partial^6 u}{\partial y_0^6}\right]+O(h^8). \qquad (5)$$

Durch Multiplikation beider Teile der Beziehung (3) mit 5 und Addition des erhaltenen Betrages zu (4) wird die Formel (1) gefunden.

Multiplizieren wir beide Teile der Beziehung (3) mit 20 und subtrahieren aus dem gefundenen Produkt (5), so erhalten wir die Formel (2).

Um das Integral der Poissonschen Gleichung, das an den auf γ' gelegenen Gitterknoten gegebene Werte annimmt, zu finden, sind n lineare algebraische Gleichungen der Art (1) oder (2) mit n Unbekannten, wo n die Anzahl der inneren Gitterknoten angibt, aufzulösen.

Wenn der Umriss γ in die geschlossene algebraische Kurve $\omega=0$, wo

$$\omega=a_{11}x+2a_{12}xy+a_{22}y^2+2a_1x+2a_2y+a_3,$$

und die möglichst eng sich an γ anschmiegt, eingeschlossen wird, so zeigt eine Berechnung, die analog derjenigen in unserer Arbeit [3] ist, dass z. B. der Formel (1) die Abschätzung

$$|\xi|\leqslant\eta+\frac{193}{151\,200}\frac{M_8h^6\,|\,\omega\,|_{\max}}{|\,a_{11}+a_{22}|}$$

entspricht. Hier ist h die Gitterlänge des Netzes, ξ — Fehler des Auflösungsnetzes, η — die genaue obere Grenze der Absolutwerte des Fehlers von u in den Grenzknoten des Netzes, M_8 — der Maximalwert der absoluten Grössen der partiellen Differentialquotienten der 8-ten Ordnung von u im von γ begrenzten Gebiet, und $|\,\omega\,|_{\max}$ — der Maximalwert von $|\omega|$ in dem von der Kurve $\omega=0$ begrenzten Gebiet.

Durch Elimination von $h^4\Delta^2 u_0$ aus den Gleichungen (3) und (4) oder (3) und (5) werden Formel erhalten, deren Anwendung keine Differentiation der Funktion $\varphi(x, y)$ erfordert. Die Restglieder in diesen Formeln sind 6-ter Kleinheitsordnung bezüglich h.

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Eingegangen
d. 28. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSICS

THE NATURE OF ORDINARY ICE

By N. J. SELJAKOV

(Communicated by S. I. Wawilow, Member of the Academy, 28. XII. 1936)

We have shown in two preceding papers (1) that ordinary ice exists in two different modifications, as α-hexagonal and β—rhombohedral ice. As mentioned, the work was carried out in one of the rooms of the meteorologic tower under the conditions of natural cooling. With the coming of spring the experiments were transferred to one of the Leningrad refrigerators where they were carried on in a room having the temperature —8° C. Here the conditions of work were much better as they were beyond, the influence of the ever changing Leningrad climate was not to be feared and they allowed to hold the pieces of ice practically during any desired period of time.

These experiments carried out under the new conditions consisted of 3 series.

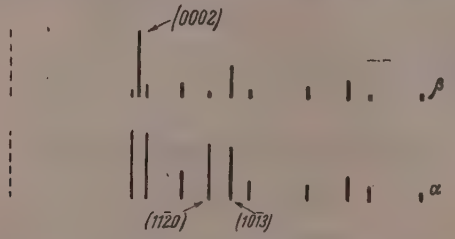
First series of experiments. The purpose of the first series of experiments was to check the results of the already published experiments carried out by us by the Laue method. For this series of experiments test pieces of ice were obtained by freezing distilled water at the temperatures of the ambient air —8 and —1° C (in the refrigerator there were rooms of 2 types, having the temperatures about —8 and —1° C). The freezing (2 cases) at —8° C gave β-ice and that at —1° C (3 cases) gave in one case β-ice and in two cases α-ice.

The β-ice was obtained in June 1936 and test pieces of it are kept up to the present time. X-ray analysis (Laue method) of the β-ice carried out after 5 months has shown that the structure of this ice had not undergone any changes.

Second series of experiments. The second series of experiments was carried out by the Deby method. For this purpose large crystals of the same test pieces of α- and β-ice on which the first series of experiments were carried out were ground to fine powder. Pressed columns 1 mm in diameter were prepared by means of a capillary tube. When both operations (the grinding of ice to powder and the pressing of it into columns) were being carried out, care was taken to prevent the thawing of ice. Since the temperature of the laboratories was —8° C, it was possible to avoid the thawing of ice. During the X-ray analysis by means of the Debye method the specimen was rotated all the time. The

roentgenogram obtained had continuous lines and this gave us the possibility to determine their relative intensity.

The experiments have shown that α- and β-ice ground to powder give quite identical roentgenograms but different from the roentgenogram of ice described by Dennison (²) (Figure) *. The position of all the lines on our roentgenogram (Debye's) coincides with the position of the lines on Dennison's roentgenogram. But the most intense line (0002) present on



Dennison's roentgenogram ist absent on ours and besides we have the ratio of the intensity of the Lines (11$\bar{2}$0) and (10$\bar{1}$3) equal to 1 : 1 while Dennison has correspondingly 1 : 5.

The calculation of these roentgenograms gave us $\frac{c}{a} = 1.62$; $c = 7.34$ Å and $a = 4.52$ Å.

Third series of experiments. The third series of experiments was carried out on ice test pieces obtained by freezing water in glass tubes about 3 mm in diameter at the temperature of the ambient air — 8° C. The measurement of the water temperature with a thermocouple and the observation of the character of crystallization **, have proved that in this case we have to do with supercooled water. X-ray analysis has shown that such ice gives a pronounced Debye's roentgenogram but in spite of the rotation of the test piece the lines are not continuous and therefore we cannot determine their intensity ***. But at the same time the presence of lines (0002) in the form of an exceptionally intense reflex could be seen on the roentgenogram. This fact pointed out that the crystals of ice had such an orientation that the plane of the basis was parallel to the axis of the test piece and consequently as well to the axis of rotation of the camera. The calculation of the roentgenogram gave $\frac{c}{a} = 1.62$; $c = 7.32$ Å and $a = 4.52$ Å.

A discussion of the results obtained and an experiment on X-ray analysis of β-ice crystal by the rotation method. In discussing the results obtained, the conditions under which Dennison's experiment was carried out should be taken into consideration. Dennison has investigated ice by freezing water in a lime glass tube (X-ray analysis in $K_\alpha M_0$) immersing it into liquid air. The water was probably supercooled then.

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* The upper part of the figure shows the position of the lines and their relative intensity according to Dennison.

** The suddennes, crystallization of all the volume.

*** All Debye's roentgenograms as well as the rotation roentgenogram which is considered later, were carried out with an iron target.

The above described experiments bring to the following conclusion. β-ice as an unstable one changes when being ground and pressed into α-ice. The units of the both modifications, α and β, are equal within 0.5 per cent. To ascertain the fact that β-ice is unstable, we carried out an X-ray analysis by the method of rotation of the β-ice crystal (the crystal that was previously investigated by the Laue method) placed so that the base plane was parallel to the axis of rotation. The roentgenogram showed the presence of a spot (0002) (the most intense one) having $d = 3.64$ Å ($\lambda = 2d \sin \theta$) (Dennison 3.67 Å) *.

For the sake of comparison it may be pointed out that the corresponding value of d for the lines (0002) in our experiments of the third series was 3.66 Å.

Papers are found in literature describing the attempts to find the full structure of ice. Among them are the works of W. H. Bragg (3) and W. H. Barnes (4).

W. H. Bragg used the data obtained by Dennison, who as far as we know investigated β-ice, and determined a structure that did not correspond to the classes of symmetry C_{3i} or C_3. The same holds true to W. H. Barnes' work. Investigating the needles of ice, W. H. Barnes had to deal with β-ice [the presence of the reflex (0002)]. At the same time the Laue's roentgenogram shown in Barnes' work, in accordance with which he determined to what class of symmetry does the ice belong, applies undoubtedly to α-ice. These facts show that up to the present time the structure of ice has not been fully and accurately defined.

John's (5) data have brougt confusion into the minds of many investigators being $\left(\frac{c}{a} = 1.40\right)$ absolutely different from those of other authors (Rinne, Gross, Dennison and Barnes: $\frac{c}{a} = 1.62$). Nordenskjold (6) has pointed out that for hexagonal lattices with $\frac{c}{a} = 1.62$, one obtains $\frac{c}{a} = 1.40$ ** when using another method of determining the indices. It is clear that the magnitude of the parameters of the lattice will be quite different. Thus in the John's work we have $c = 6.65$ Å and $a = 4.74$ Å, while in the work of Dennison we have $c = 7.35$ Å and $a = 4.50$ Å. One may easily see that when the measurements are not very accurate the position of the lines will be practically the same. But there is also a great difference consisting in the fact that when using the method of determining indices accepted by John there is no reflex with $d = 3.67$ Å. The corresponding line was probably not found. In spite of our former supposition (1), we have to assume the probability of the fact that John had to do with α-ice ***. Now the question arises, what kind of ice does exist in nature. Up to the present time we had no possibility to carry out syste-

* A complete calculation of the roentgenogram of the β-ice crystal rotation gives the same values of Bragg's angles as those determined from the Debye's roentgenogram for the second and third series of experiments.

** In our preceding note [C. R. Acad. Sci. URSS, II (XI), № 6 (92)], $\frac{c}{a}$ for β-ice is given as equal to 1.33. Repeated measurements of the same Laue's roentgenogram taking into account a large number of points gave $\frac{c}{a} = 1.36$; taking in view the necessity of an identical method of determining the indices for the α- and β-ice, the number 1.36 is to be changed into 1.58 ± 0.04 Å.

*** Unfortunately we had no possibility to get the original John's work.

matic observations but we dispose of data concerning one case. We have obtained a Laue's roentgenogram for a small piece of ice taken from one of the ponds of the Elagin Island (Leningrad). This ice proved to be β-ice.

As to the number of molecules of H_2O present in a unit of the lattice of ice, we may state that it is the same both for α- and β-ice and is equal to 4. Although we have not measured the density of ice, our assumption as to the number of molecules in a unit of the lattice of ice is quite probable since the densities of ice found by different investigators do not differ greatly.

Conclusion

All the experiments carried out by us recently and formerly bring to the following conclusions:

1. The existence of two modifications of ordinary ice, α- and β-ice, is found. α-ice belongs to one of the following classes of symmetry: D_{6h}, D_6, C_{6v} and D_{3h}; β-ice belongs to one of the two classes: C_{3i} and C_3.

2. The units of the lattices of α- and β-ice are approximately equal and are characterized by the following values of the parameters: $\frac{c}{a} = 1.60 \pm 0.02$; $c = 7.34 \pm 0.04$ Å and $a = 4.52 \pm 0.03$ Å.

3. β-ice is unstable: when being ground and pressed it changes into α-ice.

4. It is quite probable that the number of molecules in a unit of the lattices of α- and β-ice is identical and is equal to 4.

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Received
28. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSIQUE DU GLOBE

DOSAGE DE L'OZONE ATMOSPHÉRIQUE A L'ALTITUDE DE 9620 m, EFFECTUÉ PAR UN PROCÉDÉ FLUORO MÉTRIQUE

Par M. KONSTANTINOVA-SCHLESINGER

(Présenté par S. I. Wawilow, de l'Académie le 16. XII. 1936)

Les données que nous possédons actuellement sur l'épaisseur de l'ozone dans l'atmosphère à hautes altitudes sont des résultats de mesures i n d i r e c t e s, reposant sur l'étude des spectres du soleil ou du ciel bleu. Quant à des méthodes directes pour déterminer la concentration de l'ozone dans la haute atmosphère, jusqu' ici il n'en existait point. Au cours de ces dernières années j'ai élaboré d'après une idée de Prof. S. I. Wawilow un procédé fluorométrique pour déterminer la teneur en ozone, dont la sensibilité est environ 10 000 fois plus grande que celle des procédés chimiques(1). En l'appliquant nous avons réussi à effectuer pour la première fois une mesure d i r e c t e de l'épaisseur de l'ozone à l'altitude de 9620 m. L'air que nous avons analysé a été pris le 7 VII 1936 par un substratostate dont l'équipage était: capitaine Zykov en chef, et ingénieur militaire Bobykin, observateur. La prise a été faite à la hauteur maximale de l'ascension—9620 m $\pm$ 40 m au-dessus du niveau de la mer, la température étant—33° $\pm$ 1° et la pression correspondante, 221 mm. La mesure de l'altitude a été effectuée à l'aide du barographe «Metpribor» (15000 - me) № 337039; la température a été mesurée par un météorographe de sondage № 326.

L'analyse fluorométrique de cette prise d'air nous a montré que la concentration de l'ozone y est $10^{-8}.8.7$ g par 1 l de volume non réduit ou bien $10^{-7}.2.63$ g par 1 l de volume réduit. Si l'on exprime cette teneur en ozone par l'équivalente couche d'ozone cela fait 0.00404 cm par 1 km.

Comparons ces résultats avec la teneur en ozone à l'altitude de 10 km obtenue par des mesures optiques de la distribution de l'ozone (v. le tableau).

Le premier chiffre (altitude 10 km), celui de Regener, résulte de son expérience (2) relative à l'étude des photographies automatiques du spectre du soleil à différentes altitudes, qu'il a fait à l'aide d'un spectrographe élevé sur un ballon jusqu'à l'altitude de 31 km. D'après Regener l'erreur de ses évaluations ne surpasse guère 10—20%.

Les deux autres chiffres sont obtenus par Dobson et ses collaborateurs au cours de leurs études sur la distribution de l'ozone, effectuées en Suisse (Arosa) en 1932—1933 (3) et en Norvège (Tromsö) en 1935(4). La méthode appliquée dans les deux cas reposait sur l'étude de la lumière du ciel bleu zénithal dont on mesurait—durant la journée, à différentes distances zénithales du

Teneur en ozone évaluée en centimètres par un kilomètre d'air

	9.620	10.0
Konstantinova-Schlesinger (méthode fluorométrique)	0.0040	—
Regener	—	0.0048
Meetham et Dobson à Tromsö . .	—	0.0041
Gotz, Meetham et Dobson à Arosa	—	0.0061

soleil—la relation des intensités de deux longueurs d'onde, différemment absorbées par l'ozone. Ces données expérimentales permettent de calculer—en tenant compte du «Umkehreffekt» découvert par Gotz—la distribution verticale de l'ozone. Dobson a calculé, d'après ses mesures faites à Tromsö, plusieurs courbes de distribution, correspondant à des quantités totales de l'ozone variant de 0.20 à 0.30 cm. Il a montré que l'altitude de la couche de l'ozone d'épaisseur maximale est dans tous les cas la même—21 km, quant à la teneur en ozone, à l'altitude de 10 km, elle varie de 0.0036 à 0.0047 cm par 1 km. Les concentrations 0.0041 cm à Tromsö et 0.0061 cm à Arosa correspondent à l'épaisseur totale de 0.26 cm. Les courbes de distribution de l'ozone, calculées d'après les mesures à Arosa, sont plus déclives et la couche de concentration maximale de l'ozone se trouve plus haut.

Nous voyons que l'analyse fluorométrique que nous avons effectuée donne une nouvelle preuve de ce que l'ordre de la quantité d'ozone, qu'on admet à l'altitude de 10 km est juste.

Nous espérons pouvoir bientôt faire des analyses des prises d'air de plus grandes altitudes: nous pourons alors juger sur la distribution de l'ozone dans la très haute atmosphère se basant exclusivement sur des mesures directes de sa concentration.

Avant de finir je dois attirer l'attention sur le fait que cet ouvrage a pu être effectué exclusivement grâce à l'aide du colonel C. A. Prokofieff qui nous a fourni la prise d'air, aussi bien que les ballons, qui nous étaient nécessaires pour les expériences préliminaires. Je profite de cette occasion pour le remercier sincèrement.

Je suis fort obligée au Prof. S. I. Wawilow, dont l'idée et l'initiative ont été cause de cet ouvrage.

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Manuscrit reçu
le 16.XII.1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSIKALISCHE CHEMIE

DIE ISOBAREN

Von W. A. PLOTNIKOW, Mitglied der Ukrainischen Akademie, korrespondierendem Mitglied der Akademie der UdSSR

Als ich in meiner Abhandlung «Die Protonenisomerie» ([4]) den ersten Versuch machte, die Entstehung der Isobaren zu erklären, waren 8 Paar Isobaren bekannt (1928). Als ich aber im Jahre 1933 in der Abhandlung «Die Isobaren» ([6]) die von mir bemerkten Regelmässigkeiten in der Verteilung der Isobaren darlegte, war die Zahl der Isobarenpaare schon bis zu 34 gelangt, und gegenwärtig ist die Anzahl der genau mittels der massenspektroskopischen Methode festgestellten isobarischen Paare ([2], [3]) bis auf 43 ([1]) gestiegen. Man muss dabei in Betracht ziehen, dass gleichzeitig mit dem Entdecken neuer Isobare einige wegfallen, so haben sich z. B. nach den neuesten Untersuchungen manche Isotope von Zink als Hydride erwiesen.

Noch im Jahre 1924 fand Aston ([1]) folgende Regelmässigkeit in der Lage der Isobaren: in den meisten Fällen sind die Ordnungszahlen der Isobarenpaare bildenden Elemente gerade und unterscheiden sich in 2 Einheiten; die Massenzahlen sind ebenfalls gerade. Aston führt auch einige zweifellose Ausnahmen von dieser Regel an.

Wir führen hier die gegenwärtig festgestellten 43 Paare der isobarischen Elemente an.

Tabelle

		Ar	Si	Zr	Ge	Ge	Se	Se	Se	
		Ca	Cr	Ge	Se	Se	Kr	Kr	Kr	
		40	50	70	74	76	78	80	82	
	Kr	Zr	Zr	Zr						
	Sr	Mo	Mo	Mo	Mo	Mo	Rr	Ru	Pd	
	86	92	94	Ru	Ru	Ru	Pd	Pd	Cd	
				96	98	100	102	104	106	
	Pd	Cd	Cd	Cd	In	Cd	Sn	Sn		
	Cd	Sn	In	Sn	Sn	Sn	Sb	Te		
	108	112	113	114	115	116	121	122		
		Sb	Sn	Te	Te	Te	Xe	Ce		
		Te	Se	Xe	Xe	Xe	Ba	Nd		
		123	Xe	126	128	130	136	142		
			124							
Nd	Yb	W	Os	Pt	Au	Pt	Hg	Hg	Tl	Pb
Sm	Hf	Os	Pt	Hg	Hg	Hg	Tl	Pb	Pb	Bi
144	176	186	192	196	197	198	203	204	206	209

Von unten wird für jedes isobarische Paar die Massenzahl angeführt.

Wie aus der Tabelle ersichtlich ist, entsprechen 35 Paare der Regel von Aston; 8 Paare, d. h. fast der fünfte Teil aller bekannten Isobaren, bilden eine Ausnahme von dieser Regel. Bei diesen Isobaren ist der Unterschied nicht 2, wie bei den übrigen 35 Paaren, sondern bloss 1 gleich. Jedoch weisen diese 8 Isobarenpaare ihrerseits folgende Regelmässigkeit auf: die von der Aston-Regel abweichenden Isobaren kann man in Gruppen von 2 Isobarpaaren auf solche Weise verteilen, dass das der Ordnungszahl nach höhere Isobar des leichteren Paares stets ein Isotop des der Ordnungszahl nach niedrigeren Isobars des schwereren Paares darstellt. Nehmen wir z. B. aus der Tabelle die ersten zwei Paare, die eine Ausnahme von der Aston-Regel bilden: Cd—In 113 und In—Sn 115. Die Ordnungszahlen Cd (48) und In (49) unterscheiden sich bloss um 1, ebenso wie die Ordnungszahlen von In (49) und Sn (50). Jedoch enthalten beide Paare ein und dasselbe Element In aber mit veschiedenen Massenzahlen: das erste Paar enthält das Isobar mit einem Atomgewicht von 113, das zweite—115. In meinen früheren Arbeiten erkläre ich die Bildung von Isobaren durch Kernverpackung. Vier Neutrone bilden bei Ausscheidung von zwei Elektronen ein α-Teilchen:

$$4n = \overset{++}{\alpha} + 2e.$$

Das Alpha-Teilchen mit zwei positiven Ladungen bleibt im Kern und zwei Elektrone gehen in die äussere Hülle über; auf diese Weise ergibt sich ein neues Atom, bei dem die Ordnungszahl um 2 grösser ist, und die Massenzahl dieselbe bleibt, d. h. es bildet sich ein Isobar, das der Regel von Aston entspricht.

Ausser den α-Teilchen können sich im Kerne noch Deutone (*d*) befinden, deren Massenzahl um zweimal kleiner, als die der α-Teilchen ist, und die sich aus 2 Neutronen bilden:

$$2n = \overset{+}{d} + e.$$

Dieser Vorgang führt zur Bildung von Isobaren, die von der Aston-Regel abweichen, da sich die Ordnungszahlen der Isobaren in diesem Falle bloss um 1 unterscheiden.

Doch kann man die Deutonenbildung als das erste Stadium der Bildung von α-Teilchen betrachten. Daher findet in einem Isotope des gebildeten Isobars (z. B. In aus Cd) bei den günstigen Bedingungen das neue Verpakkungsprozesses statt, also bildet sich ein neues Isobarenpaar, das ebenfalls von der Aston-Regel abweicht.

Auf diese Weise bildet sich aus Cd 113 zuerst das Isobar In 113, und aus dem Isotop In 115 bildet sich das zweite Isobar Sn 115. Das Isotop In 115 unterscheidet sich vom Isotop In 113 durch die Hinzufügung von zwei neuen Neutronen. Es ist möglich, dass gerade das Erscheinen dieser neuen Neutrone die Bedingungen schafft, die zur Beendigung des Verpackungsprozesses günstig sind, d. h. zum Bilden eines neuen Deutons, das in Verbindung mit dem bereits vorher entstandenen Deuton ein α-Teilchen bilden kann. Bei solch einer zweistufigen Bildung des α-Teilchens zerfällt der Verpackungsprozess in zwei Stadien, die in zwei verschiedenen isobarischen Paaren verlaufen.

Der Prozess, der in jeder Gruppe (der zwei isobarischen Paaren), die von der Aston-Regel abweichen, stattfindet, entspricht dem Prozess, der in jedem einzelnen isobarischen Paare nach Aston-Regel vor sich geht.

Alle Isobaren, deren Ordnungszahlen sich um 1 unterscheiden, können in folgende 4 Gruppen geteilt werden

Cd — In 113	Sn — Sb 121	Au — Hg 197	Tl — Pb 206
115 In — Sn	123 Sb — Te	203 Hg — Tl	209 Pb — Bi

Auf Grund des Obendargelegten kann man sagen, dass alle Isobaren folgender Regel unterliegen: Die Ordnungszahlen der Isobaren unterscheiden sich um 2 oder um 1. Die Isobaren, deren Ordnungszahlen sich um 1 unterscheiden, bilden Gruppen zu zwei isobarischen Paaren auf solche Weise, dass das der Ordnungszahl nach höhere Isobar des leichteren Paares (In, Sb, Hg, Pb) ein Isotop des der Ordnungszahl nach niedrigeren Isobars des schwereren Paares darstellt: In 113 und 115, Sb 121 und 123, Hg 197 und 203, Pb 206 und 209.

Wenn die Isobarenbildung einen Kernverpackungsprozess darstellt, so muss diese Kernreaktion von Verlust an Masse begleitet sein. Zur Prüfung dieser Folgerung braucht man nur die mittleren Massenverluste (packing fraction) der Isobaren zu vergleichen: unserer Hypothese gemäss muss dieser Verlust an Masse beim Isobar mit der höheren Ordnungszahl grösser sein. Leider ist bloss für zwei isobarischen Paare das Atomgewicht mittels massenspektrographischer Methode genau festgestellt, nämlich für Se und Kr mit Massenzahlen ([1]) 78 und 80.

Bei den Isobaren mit der Massenzahl 78 ist der mittlere Massenverlust $\times 10^4$, für Se—8, für Kr—9.4; bei den Isobaren mit Massenzahl 80—für Selen 7.3, für Kr 9.1. Also wird unsere Schlussfolgerung vollkommen bestätigt, aber der Zahlen gibt es nur wenig, und sie sind nur annäherend festgestellt. Eine annähernde Berechnung kann man noch für das isobarische Paar Xe—Ba mit den Massennummer 136 machen. Von den verschiedenen Isotopen von Xe und Ba hat Aston den mitteleren Massenverlust bloss für Xe 134 und für Ba 138 festgestellt. Aber der Vergleich der Zahlen für verschiedene Isotopen zeigt, dass bei Änderung der Ordnungszahl um 2 der mittlere Massenverlust sich sehr wenig ändert. Daher können wir den mitteleren Massenverlust für Xe 136 annehmen, den Aston für Xe 134 erhalten hat, d. h. 5.3×10^{-24} und für Ba 136—die Zahl, die für Ba 138 gefunden wurde, d. h. 6.1×10^{-24}. Wie ersichtlich, ist auch hier bei Isobaren mit grösserer Ordnungszahl der Verlust grösser. Allerdings hat man hier mit Zahlen zu tun, für die die experimentellen Fehler sehr gross sind, doch muss man in Betracht ziehen, dass wir alle Daten benutzt haben, die wir in den von Aston veröffentlichten Resultaten finden konnten.

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Eingegangen
d. 17. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSICAL CHEMISTRY

VAPOUR TENSION OF MIXTURE OF PHOSPHORIC AND NITRIC ACIDS

By E. E. BABKIN

(Communicated by E. V. Britzke, Member of the Academy, 16. XII. 1936)

Besides mineral fertilizers, for the production of which phosphoric acid has been used for a length of time, there is a noticeable tendency in engineering of recent years to introduce this acid into other branches of industry (food-supply industry, technology of organic substances, etc.*).

At the present time, research work is being done on the application of phosphoric acid, instead of sulphuric, in the manufacture of concentrated nitric acid**. There exist various proposals on combined treatment of phosphoric and nitric acids. We have, accordingly, undertaken the task of studying the vapour tension of mixtures of these acids. To make the conditions of our work similar to those of manufacture of nitric acid, we have carried out all the investigation at an external pressure of 750 mm of mercury column.

What interested us, first of all, were the boiling points and the composition of vapour phase ternary mixtures: phosphoric acid—nitric acid—water. We were unable to find any published articles bearing upon this system. There exists only one small communication made on May 30, 1929, at a sitting of the Chemical Branch of the Russian Physico-Chemical Society by A. V. Sapožnikov and Ivčenkov [2] on the study of vapour tension of nitric acid in mixtures with phosphoric anhydride. Further communications upon this subject could not be found.

Phosphoric acid greatly resembles sulphuric acid in some of its properties (high boiling point, dehydrating properties); this enabled us to assume that the studied system should in many cases behave similarly to the system: sulphuric acid—nitric acid—water. The latter was studied by many authors, amongst whom we should mention Sapoznikov [3,4,5,6], Pascal [7], Berl and Samtleben [8].

Apparatus and Working Methods. The two binary systems limiting the studied system on two sides, i. e. phosphoric acid—water, and nitric acid—water, differ essentially as to their vapour tension. According to the data of E. V. Britzke and N. E. Pestov [9], the former, when heated from usual room temperatures to 220°C, emits only water in the form of

* An incomplete list of literature upon this subject may be found in Bull. Acad. Sci. URSS [1]. The problem of technical application of phosphoric acid in organic chemistry was strongly brought forward by Academician E. V. Britzke, more than ten years ago.

** Vide also D.R.P. 222680 and 292543.

vapour; and only beyond 220°C very smallquantities of phosphoric acid appear in the vapour phase (0.0175% H_3PO_4 at 238°C).

Vapour tension of solutions of phosphoric acids at temperatures of 20, 40, 60 and 80°C was determined by I. A. Kablukov, Honorary Member of the Academy and K. I. Zagvozdkin ([10]).

The second system emits into the vapour phase in all cases a mixture of nitric acid and water, greatly differing in its composition from the corresponding equilibrium liquid phase. Analogous facts are observed also when distilling ternary mixtures: sulphuric acid—nitric acid—water.

Such facts led us to expect considerable changes in the composition of the studied mixtures during their distillation, which would have undoubtedly distored the results obtained and, in the first place, the boiling points. We were therefore obliged to choose such methods of measurement and such apparatus as would permit of keeping the composition of mixtures during the experiments practically constant.

We chose the method of distillation of large quantities of liquids. The scheme of apparatus employed resembled the one described by Pascal.

Reagents and Methods of Analysis. For a majority of experiments commercial chemically pure nitric acid was used. In some cases pure air or carbon dioxide were blown through it to remove oxides of nitrogen. When nitric acid of concentration higher than 65% was necessary, it was prepared directly before the experiment by redistillation of chemically pure nitric acid with an equal weight of concentrated sulphuric acid.

Reacting phosphoric acid, was subjected to threefold crystallization and gave a product containing 87% of H_3PO_4. No impurities, excepting water, were traced in it on analysis. The mixtures of acids were prepared on technochemical scales with an accuracy to within 0.2 grams. All samples of mixtures were subjected to a chemical analysis before and after the experiment to verify their exact composition*.

First of all, the general acidity of the mixture was determined by simple titration with methyl-orange, and then the content of phosphoric acid was ascertained by precipitation according to Schmitz. Nitric acid and samples of distillates were titrated by alkalies of suitable normality. The mean of five determinations was taken as the content of nitric acid in the distillate.

Partial Tension of Nitric Acid Vapour over Boiling Mixtures: H_3PO_4—HNO_3—H_2O. Some results of our determinations are given in the table. The partial tension of nitric acid vapour was calculated from the percentage content of nitric acid in the distillate, as obtained by analysis.

In order to represent the variations of vapour tension more graphically, the data given in the table have been plotted in curves of Fig. 1. Partial tensions have been plotted along the axis of ordinates and additions of phosphoric acid (in percentages by weight) along that of abscissae.

The curves of mixtures containing up to 30% of HNO_3 have not been plotted to the end, as no further experimental data were available. For the same reason, the curve of mixtures containing 31.0% of HNO_3 has been continued by a dotted line. The curve of mixtures containig over 80% of HNO_3 have been traced partly by graphical extrapolation. It should be noted that in this region the method of measurement and the analysis give insufficiently exact results because of partial decomposition of nitric acid.

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* The analyst A. D. Griničenko took part in the analytical part of the investigation.

Table

No	Composition of mixture by wt. in %			Boiling point °C	Content of HNO_3 in vapour by wt. in %	Partial tension of nitric acid vapour over the mixture in mm of mercury column
	H_3PO_4	HNO_3	H_2O			
1	28.74	6.72	64.54	105.0	0.64	1.34
2	60.85	5.93	33.22	121.0	6.56	14.4
3	69.50	5.85	24.65	133.0	24.24	61.2
4	10.07	11.46	78.47	103.0	0.60	1.3
5	39.83	10.86	49.31	112.0	2.36	5.0
6	49.82	11.54	38.64	118.0	8.60	19.0
7	72.36	11.48	16.16	138·0	63.79	244.1
8	21.32	20.08	58.60	111.0	3.79	8.1
9	41.63	19.09	39.28	118.5	20.23	49.3
10	51.71	19.09	29.20	125·0	45.33	139.8
11	62.71	17.81	19.48	133.0	69.04	284.0
12	10.73	29.51	59.75	112.0	7.21	15.9
13	20.79	31.04	48.17	116.0	15.31	35.9
14	30.77	31.09	38.14	119.5	34.00	93.7
15	53.25	31.02	15.73	121.0	85.50	458.2
16	9.57	39.87	50.56	116.0	20.79	50.9
17	32.41	39.95	27.64	124.0	65.75	258.6
18	20.85	49.08	30.07	122.0	65.62	257.5
19	33.56	49.22	17.22	119.0	91.38	548.8
20	10.75	60.16	29.09	121.0	68.88	282.8
21	20.83	59.93	19.24	117.0	91.17	545.0
22	24.63	60.09	15.28	113.0	96.22	641.8
23	5.90	65.08	29.02	120.5	69.04	284.0
24	9.98	65.48	24.54	119.0	78.90	377.0
25	11.43	69.31	19.26	114.0	90.13	527.4
26	14.82	68.60	16.58	111.0	95.76	632.0
27	5.71	74.71	19.58	113.0	90·73	538.0
28	11.42	74.18	14.40	108.0	96.10	639.0

It will bee seen from a close examination of the Table and of the curves of partial tension of the system (Fig. 1) that when mixtures containing over 39% of water are distilled, a solution of nitric acid very rich in water enters the receiver. The percentage content of HNO_3 in it is less than that of HNO_3 in the original mixture. The tension of water vapour in this region of mixtures is very great.

This should be readily understood, if we take into account that nitric acid is able to form hydrates with water and that, according to Sapožnikov's opinion (4), one particle of water can fix several particles of HNO_3. Phosphoric acid added to the solution of nitric acid fixes water and frees particles of nitric acid, thus increasing the partial tension of HNO_3 in the mixture. However, in mixtures rich in water (more than 39%) such a

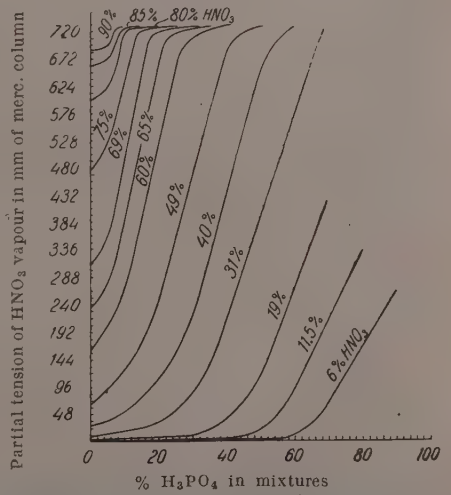

Fig. 1. Partial tension of HNO_3 vapour over boiling mixtures H_3PO_4—HNO_3—H_2O, containing an equal quantity of HNO_3

dehydrator as phosphoric acid is unable to fix water to such an extent as to give the same or a higher percentage of HNO_3 in the distillate as compared with that in the original mixture.

A diminution of the percentage of water in mixtures below 39% leads to a marked increase of the content of nitric acid in the distillate. Even small additions of phosphoric acid produce a distillate richer in nitric acid than the original mixture. Very soon the content of HNO_3 in the vapour exceeds 50%. However, after the partial tension has reached a certain limit (about 680 mm), there begins a rapid slowing down of the increase of HNO_3 content

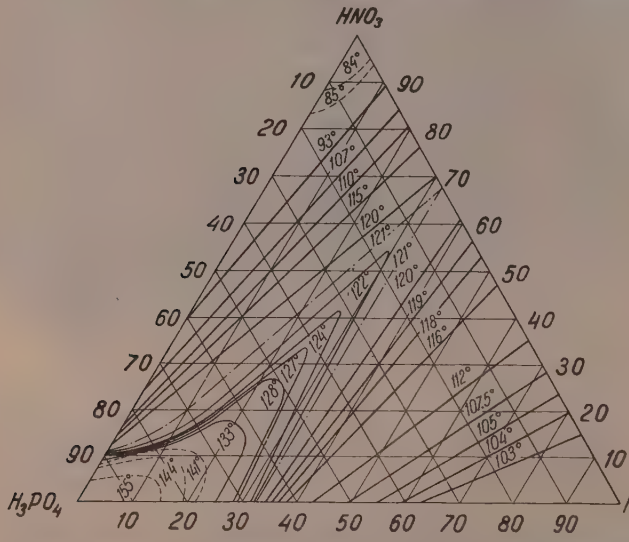

Fig. 2. Boiling point isotherms of mixtures H_3PO_4 — HNO_3—H_2O under pressure 730 mm merc. column

in the vapour. It is impossible to avoid partial decomposition of nitric acid in this region. Oxides of nitrogen, oxygen and water vapour evolved greatly lower the partial tension of nitric acid vapour and the true boiling point. When data pertaining to this region of the system are examined, these circumstances should be taken into account.

The Boiling Points of Ternary Mixtures: H_3P_4O—HNO_3—H_2O. Ebullition isotherms of ternary mixtures were plotted in Gibbs triangle from data obtained during the investigation (Fig. 2).

Isotherms 84, 85, 141, 144 and 155° are shown by dotted lines, as experimental data available for them were scanty. However, the configuration of adjacent isotherms gives the author confidence that they are traced with an exactitude sufficient for purposes of technical calculation.

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Received
16 XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

CHEMISTRY

CONDUCTANCE OF ELECTROLYTIC SYSTEMS

By M. A. KLOČKO

(Communicated by N. S. Kurnakov, Member of the Academy, 21. XI. 1936)

§ 1. At the present time, there exists no theory of electrolytic conductance that could at least qualitatively explain the change of conductance in systems due to change in concentration and temperature. The old theory of Arrhenius, as well as the latest electrolytic theories, are applicable to greatly diluted solutions, i. e. to a very limited part of the conducting system. A large part of experimental material on electrolytic conductance relates to such diluted solutions, while in those cases where more concentrated solutions were investigated, the results were worked up and interpreted from the standpoint of the above-mentioned theories, which led to an one-sided understanding of the phenomenon under investigation. Owing to the quantitative predominance of the solvent in the systems studied the solvent was believed to possess certain special properties, as a «dissociating force», by means of which it was attempted to explain the dissociation of electrolytes into ions; the individual properties of the electrolyte, in particular its conductance, were hardly at all taken into account.

The few investigators that regarded solutions as systems consisting of equivalent (as objects to be studied) components, could not find a correct interpretation of the isotherms of conductance and were inclined to take the composition corresponding to any remarkable point on these isotherms (maximum, minimum. point of inflection) as a chemical compound, independently of the fact whether this compositions was constant or variable or even the remarkable point disappeared altogether when the equilibrium conditions of the system were disturbed.

§ 2. The best way of studying the change of properties of chemical systems in equilibrium in connection with the change of their composition is to apply the theory of physico-chemical analysis, whose foundations were laid by Member of the Academy N. S. Kournakov (1).

In the physico-chemical analysis, the systems and their properties are studied «in motion», i. e. the change of properties is investigated in connection with the change of the conditions of equilibrium (concentration of components, temperature, pressure, etc.), throughout the entire period of existence of the system in question. Furthermore, the properties of the system are studied in their mutual connection, allowance being made for the influence of the properties of the initial components.

Finally, the theory of the singular elements of the chemical diagram, which forms the backbone of physico-chemical analysis and according to which a chemical compound formed in the system manifests its individuality on the diagram of any property concerning the system and measured within the limits of existence of the compound in question, unites in one point of view systems consisting of components with quite different (sometimes opposite) chemical properties and functions.

§ 3. Applying the principles of physico-chemical analysis to electrolytic systems and starting from the modern ideas regarding the properties and structure of electrolyte in an individual state, we can make some statements that will enable us to predict the change of conductance in liquid systems in connection with the change of their concentration and temperature.

1. The system conducts the current (electrolytically) if one (or several) of its components or products of their mutual chemical action conducts the current individually.

The role of non-conducting or poorly conducting components of the system—besides the possibility of forming conducting compounds—consists in converting the electrolyte into a liquid state («dissolution»), if under the conditions of the experiment it is in a solid aggregate state in which case its ions acquire a greater mobility. Furthermore the non-conducting component decreases the conductance of the system in proportion to its concentration.

2. The formation of a conducting compound from poorly conducting (or practically non-conducting) components is accompanied by the appearance of ion bonds and the increase of electric asymmetry of the molecules of the newly formed electrolyte as compared with that of the molecules of the initial components.

3. The conductance of the system depends on 1) the number of ions within it, 2) their mobility.

The number of ions, almost over the whole range of concentrations of the system, varies directly as the concentration of the electrolyte (or electrolytes, if there are several). The change in the mobility of the ions is chiefly due to the change in viscosity.

4. The change in temperature of the system affects the conductance chiefly owing to the change in viscosity (and consequently, the mobility of the ions).

In the region of temperature where a thermic dissociation of the electrolyte into non-conducting components takes place, the change of temperature is connected with the change in the number of ions.

5. The influence of electrostatic forces existing between the ions of the system upon the conductance values manifests itself at such (very small) concentrations of the electrolyte where in the volume element exceeding several times the volume of anion there may arise an electric field caused by the charges of the neighbouring ions, which may affect the mobility of the ion. The greater the concentration of the electrolyte, the greater the probability of absence of a considerably electric field around each ion and, consequently the weaker the influence of the inner electrostatic forces on the mobility of the ions.

§ 4. On the basis of these statements, let us try to find out the general aspect of the isotherms and polytherms of conductance of systems. In consequence of the decrease of viscosity with an increase of temperature, the polytherms of conductance of systems, including those having but one component, must generally have a positive temperature coefficient. In the region of temperatures where the number of ions in the system decreases due to the

increase of the thermic dissociation of the electrolyte with the increase of temperature, the polytherm of conductance may, on passing beyond the maximum, change its positive temperature coefficient for a negative one. In the latter case, the decrease in the number of ions due to thermic dissociation affects the conductance value to a greater extent than the increase of the mobility of the ions due to a decrease of viscosity.

If there is only one conducting component present in the system, the isotherm of conductance of such a system should smoothly slope down from the conductance value for this component to the zero value (or a very small value) for another component. In a ternary system represented by a trihedral prism with the triangle of composition at its base, the isothermal surface will slope down from the value of this property for the conducting component to the corresponding values (approximating zero) for the two other components, i. e. to two points situated very near to base of the prism and corresponding to the composition of these two components.

If all the components of the system are conductive, then, in the case of a binary system, the isotherm will smoothly slope down from the conductance value of the more conductive component to that of the less conductive one.

In a ternary system, we shall have in this case a smooth surface joining the value of the given property for the three components.

If a conductive compound is formed out of poorly conducting (or practically non-conducting) components, the isotherm of conductance of a binary system must have a maximum corresponding to one and the same composition (abscissa), independently of the temperature, over the whole range of temperatures, at which this compound exists (a singular maximum if there is no dissociation of the electrolyte into non-conducting components).

In a ternary system, there must exist a singular edge beginnig with a maximum on the lateral side of the triangular prism if the conducting compound is formed of two of the three components, and passing smoothly down to the conductance value (in this case approaching zero) of the third component.

Thus, there are two principal types of isotherms of conductance (for a binary system).

1. The curve connects the conductance values for the initial components if one or both initial components are conductive.

2. The curve shows a maximum whose abscissa is not affected by a change in temperature if the chemical compound formed from the components is conductive.

Combinations of these two types are possible.

Changes of viscosity may produce a marked change in the isotherm of conductance. The viscosity values of the chemical compound formed in the system being high, the mobility of the ions of this compound is so small, as compared with their mobility for other compositions of the system, that the maximum on the isotherm of conductance is smoothed down, disappears and—as is often the case by lower temperatures—is converted into a minimum.

§ 5. Up to now, the influence of viscosity on conductance has not been ascertained quantitatively. At temperatures removed from those causing a considerable disintegration of the electrolytes into non-conducting constituents, this relation may be expressed, for systems of one or many components, by the equation:

$$\overset{n}{\varkappa\eta} = \text{const}$$

where $\varkappa$ is the numerical value of the conductance of the system at a given temperature, η is the corresponding viscosity value and n is a positive number whose value ranges for many substances and systems from 0.7 to 1.5 (²).

This equation is applicable in the case of changing the temperature of the compositions. Inasmuch as u in the above-mentioned formula approximates unity, we can multiply the conductance value for various compositions and temperatures of the system by the corresponding values of viscosity, and thus, eliminating the influence of the mobility of ions on conductance values, we may get an idea of the change of the number of ions in the system with the change in concentration or temperature (or in both together).

The polytherms of conductance in regions removed from the disintegration of electrolytes, corrected in this way, will run parallel to the axis of temperatures (it is true, though, that one must take into account the change of specific volume with the change of temperature); the corrected isotherms will have the form shown in the preceding paragraph.

In the case of temperatures approaching those at which the system starts freesing or in the case of high concentrations of the components whose viscosity is considerable, the isotherm $\varkappa \cdot \eta - \varepsilon$ (where ε is the composition) follows in these portions of the diagram, chiefly the course of the isotherm of viscosity. However, when the temperature increases, this «excessive» influence of viscosity decreases, and $\varkappa \cdot \eta - \varepsilon$ isotherm assumes the form described in § 4.

§ 6. Let us examine the form of the isotherms and polytherms of conductance for some systems.

a) The System «Salt-Water». Salts are good conductors in an individual liquid state. Water converts salt into a liquid state, which imparts a greater mobility to its ions in comparison with their mobility in a solid state. In this case, water cannot increase the number of ions—the increase of the quantity of water decreases the conductance, which, on the other hand, increases as the concentration of the salt increases. The fact of reaching a maximum and the further decrease of conductance with the increase of the concentration of the salt is due entirely to viscosity: this maximum is reached for various systems at such concentrations where viscosity considerably increases with concentration. The isotherm $\varkappa \cdot \eta - \varepsilon$ corrected for viscositysmoothly rises from the value $\varkappa \cdot \eta$ for pure water up to the value for the most concentrated solutions of the salt or up to the same for pure salt, if there is a possibility of investigating conductance and viscosity from 0 to 100% of salt.

b) Substances poorly conducting in individual state, yielding together with water good conductive solutions form with the water chemical compounds that satisfy the conditions set forth in 2 of § 3. In this case there are maxima which correspond to the composition of these compounds, which become particularly marked when the influence of the viscosity of the system is taken into account (sulphuric acid—water and others).

c) Systems of Molten Salts. In this case, both components of the binary system are conductive, and the isotherm of conductance links the values of this property for both components. Maxima are unlikely as each new compound must consist of the same ions as the components, and consequently, the number of ions will not change; there are no reasons to expect changes of the mobility of the ions which could be caused by changes of viscosity. If, however, a clearly expressed complex compound is formed, one might rather expect a minimum of conductance to occur, due to the formation of a smaller number of ions, which is not always compensated by an increas-

ed valency of the newly formed ions. The majority of the salt couples so far investigated do not show any considerable minima or maxima.

d) Non-Aqueous Solutions. It occurs very frequently in these systems that compounds that are good conductors are formed from poorly conducting components, these compounds being characterzed by a maximum whose composition is not affected by temperature (benzoic acid—aniline) (3).

However, in the majority of cases, this maximum passes into a minimum under the influence of a sharp maximum of internal friction. Then the maximum may be detected on the isotherm of conductance corrected for viscosity (methyl—aniline—allyl mustard oil) (4). Hantzsch has found that the dissolution of a number of substances in sulphuric acid does not increase their conductance (5), while M. Usanovich, who found that in the system «dimethyl sulphate—sulphuric acid» the conductance decreases from the second component to the first, affirms that we have in this case a new type of isotherms of conductance (6).

In the light of the statements developed in the present article, the isotherms of conductance of the systems $(CH_3)_2SO_4$—H_2SO_4 do not essentially differ from those of the systems «salt—water». In both cases, the conductance decreases from 100% of the conducting susbstance to 100% of the non-conducting one, the water «diluting» conductance of the salt in the same manner as $(CH_3)_2\ SO_4$ does the conductance of sulphuric acid. The similarity becomes sitll more evident if we take the systems $AgTl\ (NO_3)_2$—H_2O(7) or $AgNH_4(NO_3)_2$—H_2O(8), where at 90—100° both components—salt and water—are in a liquid state.

The task that confronts us now is to revise the available experimental material on electrolytic systems in the light of the views set forth above, supplement it with new experimental investigations, and pass from qualitative («topological») relations to quantitative («metrical») ones.

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Received
21 XI. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

GEOLOGY

ON THE AGE OF OLD ROCKS AT ISACHKOVSKI HILL

By I. LOBANOV and A. JUNGERMAN

(Communicated by A. A. Borissiak, Member of the Academy, 19. XI. 1936)

Isachkovski Hill has for some time past engaged the attention of geologists owing to an outcrop of crystalline diabase lying amid vast spaces of sedimentary deposits and as much as 100 km distant (northwards) from the Ukraine Crystalline Plateau. While studying the diabase, the investigators also gave a great deal of attention to the accompanying rocks.

In spite of a rather considerable volume of literature dealing with Isachkovski Hill, the old rocks associated with the diabase have so far been considered non-fossiliferous. They offer at first sight a highly intricate variety, and this accounts for the failure of the investigators to determine their true nature up to such time as a salt dome was discovered at Isachki.

In the outcrops, these rocks are represented by olive-green, olive-yellow, ash-grey, dark grey, etc., marly clays with inclusions of large quantities of angular fragments of shales of a similar nature, and also of fine-grained recrystallized and marmoraceous grey or dark limestones or sandstones, which accounts for the brecciated character of the rock. In some of the outcrops, extended by clearing, these rocks are represented by similar argillaceo-marly shales with thin limestone streaks strongly corrugated into minute folds, while in places the shales crop out simply in occasional patches. All the ancient rocks are evidently argillaceous marly shales with streaks of limestones, which had, at the time of the formation of the salt dome, been subjected to tectonic processes, strongly fractured and wrinkled into minute folds, and subsequently weathered. The investigators of Isachkovski Hill, failing to find any direct indication as to the age and genesis of the rocks in question, put forward a number of contradictory hypotheses, which did little to elucidate the problem. Thus, Gurov took these rocks to be partly glacial, and partly marine deposits; Armashevski described them as products of the weathering and destruction of diabases; to Morozovich, they are tertiary tufites redeposited by quarternary lakes, etc.

Shatski, after studying the literature available on the subject, has come to the conclusion that these rocks are possibly Pre-Mesozoic marine strata forced up by the salt dome.

The sedimentary origin of these rocks is irrefutably proved by the occurrence therein of fossil fauna, first discovered opposite Grebistche village

by Peysik, geologist of Glavneft, and subsequently found in large quanti ties by I. N. Lobanov, of the Institute of Geology of Kharkov University, in a ravine just opposite the Isachki cemetery.

The ravine in question exhibits the following section:

1. Soil . . . . . 0.35 m.
2. Pale yellow-greyish loess . . . . . 3.45 m.
3. Yellowish-brown boulder-clay . . . . . 3.80 m.
4. Red-brown clay . . . . . 1.80 m.
5. Pinkish, friable micro-granular marly-calcareous rock with inclusions of a compact, black or gray crystalline marmoraceous limestone, in the shape of angular fragments of weathered diabase blocks, or of fragments of grey-green argillaceous shales . . . . . 1.40 m.
6. Olive green, or olive-yellow, rock, consisting in the main of an agrillaceo-marly mass, with inclusions of angular fragments of grey argillaceous shales, of a dark crystalline limestone, and of a grey stratified crystalline limestone containing remnants of fauna. Apparent thickness . . . . . 2.50 m.
7. Near the mouth, seam 6 is abruptly replaced, in a horizontal direction, by cherry-coloured clays abounding in friable calcareous pockets and concretions. The upper limiting surface of the seam 7 has a northerly dip, i. e. the cherry-coloured clays dissappear below olive-green brecciated rocks. Apparent thickness . . . 4.50 m.
8. A greenish-gray diabase occurs as rubble, which crops out along the very thalweg from under the cherry-coloured clays.

The fossils found in seam 6 are naturally of great interest, since they furnish valuable indications as to the age of the ancient rocks at Isachki.

The fossils are of a mixed character, consisting of *Pelecypoda*, *Brachiopoda* and *Gastropoda*, preserved as nuclei and shells, and found in a very poor state of preservation. An extreme paucity of species is also to be noted. As to microfauna, none has been detected in the thin sections so far made. According to a preliminary determination of *Pelecypoda*, the following forms are represented, which are also to be found in the Permian deposits of the Europian part of the USSR, viz. *Clidophoras*, cf. *Pallasi* Verneil., *Nayadites castor* Eichw., *Nayadites Fisheri* Amalizky.

No sweeping conclusions would be justified in view of the small number of the species so far determined. As, however, such forms as have been identified are characteristic of our Permian deposits, we may conclude that the brecciated rocks of Isachkovski Hill are to be classed as Permian.

Moreover, the data stated above would warrant the assumption that the Permian deposits occurring in the Donetz Basin extend right into the North Ukraine basin.

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Received
19. XI. 1936.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MICROBIOLOGY

AZOTOBACTER AS A SOURCE OF NITROGENOUS NOURISHMENT FOR THE HIGHER PLANTS

By T. T. DEMIDENKO and F. F. TIMOFEIEVA

(Communicated by A. A. Richter, Member of the Academy, 7. XII. 1936)

The azotobacter from the time of its isolation by Beijerinck has attracted the attention of investigators as an energetic fixer of molecular nitrogen. The numerous investigations of this microorganism have shown the possibility of utilizing it as a nitrogen collector for the nourishment of the higher plants. This fixation of nitrogen, being connected with the process of ammonia synthesis, is regarded as an exothermic process, in which molecular nitrogen is reduced to ammonia which is further converted into amino-acids, from which the protein molecules are built.

S. P. Kostychev (1) showed that part of the ammonia is discharged from the cells into the surrounding medium and may become immediately available to the plant—which refuted the idea that had existed before about the necessity of mineralization of the cells after their death in order that plants may be nurished.

It has been found by the investigations in Academician A. A. Richter's laboratory into the chemism of the fixing of molecular nitrogen by the azotobacter, that ammonia is the first product in the process of assimilating free nitrogen, which may be utilized immediately for synthetic purpose by the higher plants. Inasmuch as the azotobacter is a microorganism which lives independently in the soil, it follows that in order that it may develop and succeed in fixing nitrogen, there must exist a number of conditions, namely: favourable aeration (3), cultivation of the soil, which is connected with extensive availability of oxygen within the range of azotobacter's acitivity (4), optimum temperature (5), humidity of the soil (6) and a sufficiency of energy-supplying material in the nutrient medium.

Under natural conditions, the products of the decay of vegetable cells and the rhizosphere of the higher plants, containing the root excretions, constitute such a material (7,8).

Not all plants are capable of furnishing the azotobacter with nutrient material; if a plant in whose rhizosphere there is observed an abundant growth of the azotobacter be replaced by another whose root excretions are not favourable to the azotobacter, the ability of the last to assimilate nitrogen is diminished, and it gradually dies out. Thus, the azotobacter requires,

for its development, a selection of plants whose rhizosphere serve it as a source of nourishment.

Sheloumova ([9],[10]) obtained quite effective results with tobacco, maize, the sugar beet and the potato. The studies available ([11], [12], [13]) on the utilization of the azotobacter as the biological factor in the fixation of molecular nitrogen for the nourishment of the higher plants do not give an uniform picture of fixation.

Together with positive data, we have a number of studies in which the assimilation of the nitrogen of the azotobacter by cultivated plants gave no marked effect ([13], [14], [15]), which would seem to be the cause for a cautious attitude toward depending on the azotobacter in practice.

For the purpose of extending and supplementing the present data on the fixation of nitrogen by the azotobacter, we carried out, in 1935, a series of experiments in utilizing the azobobacter as a source of nitrogenous nourishment.

Experiments

The experiments were conducted on the rich black soil of the Voronezh region. The soil was first tested for the presence in it of the azotobacter. An analysis showed that it was not present in the soil. The sugar beet was taken as the experimental plant. The principle of the experiments consisted in comparing the action of the nitrogen of the azotobacter with that of organic nitrogen, introduced in the form of asparagin and urea, and mineral nitrogen. The mineral nitrogen was supplied in the form of chemically pure salt $NaNO_3$ in the amount of 1g of active principle. The organic nitrogen was likewise supplied in the amount of 1 g per pot, into which was put 5 kg of absolutely dry soil. As a general background, KCl in the amount of 0.75 g K_2O to the pot and phosphorus in the form of precipitate in the amount of 0.5 P_2O_5 to the pot were used.

The azotobacter was introduced in the form of slime, 50 cm^3 to the pot. The first introduction was together with the seed at the time of sowing; the second was made after a month, when the sugar beet had sufficiently developed.

The experiments with the beet were carried out under quite favourable external conditions; through the course of the summer, the pots stood in a box, where the temperature was high; they were watered three times a day; the moisture was maintained at 50 per cent of the absolute moisture of the soil. The actual acidity of the soil was brought to 7.2 pH.

Let us examine the results with sugar beets fertilized with mineral nitrogen, organic nitrogen and the azotobacter as a source of nitrogen (Table 1).

In analysing the yields, we see that the plants utilized mineral nitrogen applied in the form of $NaNO_3$ very well and that the yield was considerably increased in comparison with the control pots, likewise that organic nitrogen was quite advantageous, urea giving a better yield than asparagin.

As regards the variants with the azotobacter, it is evident that biological nitrogen, fixed by bacteria, is assimilated by the beet and that a higher yield results. A higher yield of sugar beet results also when PK is applied along with the azotobacter, which yet proves that the sugar beet does use nitrogen fixed by the azotobacter. The introduction into the soil of carbohydrates in the quantity of 1 g to 5 kg of soil did not give any marked positive effect because, it would seem, the azotobacter obtained a sufficient quantity of energy-supplying matter from the beet itself. If the nitrogen absorbed by the plants during the vegetative period be computed, a quite simple rela-

Table 1

Influence of the Azotobacter on the Yield and Composition of the Sugar Beet

Serial No	Details of the experiment	Yield		Total nitrogen in the crop (root and leaves) in grams	Nitrogen left in the soil after the harvest in percentages
		The root in grams	In percentages		
1	Control (soil without fertilizer) . .	90.96	100.0	0.2013	0.259
2	Soil + $NaNO_3$	165.38	181.7	0.8283	0.296
3	» + PK	136.48	150.0	0.2414	0.316
4	» + KPN	496.48	545.5	1.4590	0.404
5	» + PK + aspargin	285.94	314.3	0.7084	0.256
6	» + PK + urea	376.54	413.9	1.0962	0.278
7	» + azotobacter	154.42	169.7	0.4782	0.295
8	» + PK + azotobacter	160.14	176.0	0.5404	0.375
9	» + PK + glucose + azotobacter .	124.10	136.4	0.3811	0.445
10	» + PK + mannite + azotobacter	123.2	135.5	0.4534	0.336

tion becomes apparent: the greater the amount of nitrogen in the soil, the more of it the plants absorb during their vegetative life.

Besides the data on the nitrogen found in the crop itself, we have data from analysis of the soil nitrogen, the dynamics of which is as follows: in the control, the figures for nitrogen are 0.296 per cent and for the azotobacter 0.295 per cent; thus, the experiment with the azotobacter shows that a part of the nitrogen in all the variants in which this microorganism was present remains unused, for the reason, it would seem, that it is in an organic form hard to assimilate for the beet. This is especially apparent in the pots that received carbohydrates and azotobacter.

Besides these greenhouse experiments we conducted others in the field in 1936, in order to study the effectiveness of the azotogen under the sugar beet. The plots contained 20 m^2 each, there being 4 plots in each variant. Inasmuch as the summer of this year was quite hot and dry (during the course of the experiment, the precipitation at the station was 150 mm), it was necessary to water the beets in order to maintain sufficient moisture in the soil for their growth. The plots were watered 5 times during the summer, 30 mm each time. Azotogen was applied twice: together with the seed, when sowing, and again when four or five pair of leaves had appeared and when the beet was capable of supplying the azotobacter with the energizing material required for their vital acitivities.

To illustrate the effectiveness of azotogen, we give the data on the yield and an analysis of the field experiment (Table 2). Although the expirements were carried on only during one year, they are so significant and so in accord with the greenhouse data that we consider it advisable to present them.

In this experiments, 120 kg of the active principle of the fertilizer (PK and NPK) were applied to each of the plots. From the data shown it is apparent that quite a high yield was got from all the plots; this is explained by the fact that the summer of this year was very hot and very rich in sunlight, so that with frequent stirring of the soil and watering the beets developed quite rapidly and luxuriantly, giving a high yield. The first place in productiveness was taken by beets grown with full fertilization (450 centers

Table 2

Influence of Azotogen on the Yield and Composition of Sugar Beet

Serial No	Details of the experiment	Yield of roots		Percentage of sugar	Percentage of total nitrogen in roots	Percentage of total nitrogen in the soil after harvest
		In centners per hectare	In percentages			
1	Soil (without fertilizer)	329	100.0	15.4	0.2375	0.2792
2	» + azotogen	350	106.3	15.8	0.2537	0.2865
3	» + PK	362	110.0	16.1	0.2435	0.2747
4	» + PK + azotogen .	375	113.9	16.5	0.2631	0.2931
5	» + PKN	456	138.6	17.1	0.3634	0.3014

per hectare), the second, by those fertilized with PK plus azotogen, the third those fertilized with PK, and fourth, those with azotogen alone.

The highest content of sugar was in beets that received full fertilization, since this gave them the most favourable conditions for growth. The second place was taken by the beets that received PK plus azotogen, third place by those that received PK and the fourth, azotogen alone.

The following may be regarded as the rule with respect to the amount of nitrogen absorbed by plants during their vegetative period: if the plant has at its disposal a sufficient quantity of nitrogen, it absorbes nitrogen in considerably greater quantity if conditions are otherwise. The content of nitrogen in the soil fertilized with azotogen increased during the vegetative period, owing to the intensified activity of the azotobacter that developed under the presence of the energizing material produced by the roots of the beets in the soil solution.

In conclusion, in evaluating the azotobacter and azotogen as factors for enriching the soil by means of biological nitrogen, the following may be said.

1) Azotobacter and azotogen may be utilized in field and greenhouse conditions as nitrogenous fertilizers under the sugar beet, and give a considerable increase in the yield.

2) Azotogen, besides increasing the yield of beets, promotes the accumulation of sugar, increasing both the absolute and the relative content of sugar in the root.

3) Besides increasing the yield of beets, asotogen enriches the soil with nitrogen and this exerts a favourable influence on the crop of the following year.

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Received
7. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

MICROBIOLOGY

THE ROLE OF STRAW AS A SOURCE OF CARBOHYDRATES FOR NODULE BACTERIA

By T. T. DEMIDENKO and E. F. TIMOFEIEVA

(Communicated by A. A. Richter, Member of the Academy, 7. XII. 1936)

It is well known that the nodule bacteria, which penetrate into the roots of leguminous plants, facilitate the nourishment of the latter with atmospheric nitrogen. There is thus created a symbiosis between the plants and the bacteria, the latter at the beginning of their development utilizing the carbohydrates from the host plant.

After the appearance of the sprouts, the young plants store up but little carbohydrate, and the development of the bacteria goes slowly, inasmuch as, it would seem, the bacteria develop weakly without the presence of a sufficient quantity of easily assimilable carbohydrates.

In order to test this supposition and study the influence of carbohydrates on the development of nodule bacteria and, consequently, on the yield of legumes, we devised experiments with the soy and pea, in which these were planted in rich black soil, sterile as regards nodule bacteria, dextrin, saccharose and glucose being introduced into the soil to the amount of 1 g per 5 kg of absolutely dry soil.

The following very interesting phenological observations are to be noted: in all the pots supplied with energizing matter but without inoculation of the soil with the bacteria, the soy and peas developed rather weakly. When the soil was inoculated with nodule bacteria, both soy and pea quickly started to grow—leaves and stalks were healthy, succulent, and the number of seeds produced was much greater than on the uninoculated soil. The development of plants supplied with carbohydrates and nodule bacteria was in the following order: best of all was dextrin; second, with saccharose; and finally, with glucose.

For simplicity's sake, we present (in Table 1) the data for the soy only, since the pea responded to the introduction of carbohydrates in exactly the same way as the soy.

From these data, the following conclusions can be drawn.

1) Both in respect to grain and straw, soy and pea gave but small yields in the control pots, since they had an insufficient supply of nutritive substances.

Table 1

Serial No	Detail of experiment	General yield (grain+straw)		Amount of nitrogen absorbed in milligrams		
		In grams per pot	In percentages	In above-ground parts	In roots	In total yield
	A. Without inoculation					
1	Control (without fertilizer) .	13.79	100.00	208.3	19.6	227.9
2	Soil + PK	16.25	122.01	216.9	17.1	233.0
3	» + PKN	33.76	241.91	—	—	—
4	» + PK + glucose	13.15	95.35	172.1	12.5	184.6
5	» + PK + saccharose . . .	12.24	88.75	215.3	11.3	226.6
6	» + PK + dextrine	11.82	85.71	172.5	13.3	185.8
	B. Inoculation with nodule bacteria					
7	Soil (without fertilizer) . . .	19.29	139.81	545.9	39.18	585.0
8	» + PK	30.69	222.50	803.0	73.4	876.4
9	» + PK + glucose	42.95	311.45	1 289.9	63.16	1 353.0
10	» + PK + saccharose . . .	43.20	313.27	1 131.4	55.82	1 187.2
11	» + PK + dextrine	53.48	387.81	1 258.9	84.5	1 343.4

2) With the introduction of PK, although the yield of soy and pea increased somewhat, still, the effectiveness of these fertilizers was not so well defined because the plants had lacked nitrogen, which under the conditions of the experiments had been present in minimum quantity in the black soil.

3) With PK present, when the soil was inoculated with nodule bacteria, the yield of the plants rose sharply, giving about the same increase in yield as when full fertilization was used.

4) Carbohydrates with PK invariably diminished the yield of soy and pea, since, under the influence of the energizing material the microflora vigorously develops and transforms the mobile substances into forms hard to assimilate.

5) The highest effect on yield was obtained from dextrin, while saccharose and glucose gave nearly identical results.

6) The greatest increase of root mass was obtained from carbohydrates with the presence of bacteria; next came full fertilization, and lastly PK with carbohydrates, without inoculation.

7) The amount of nodules in the inoculated pots was invariably quite great; the maximum quantity was obtained from dextrin, while the results from saccharose and glucose were nearly identical.

8) Both absolute and relative nitrogen content in the overground and underground parts of the plants are increased when the soil is inoculated with nodule bacteria. If the amount of nitrogen assimilated by the plants be compared, it is shown that the bacteria utilized the energizing material productively in the case of glucose, then in the case of dextrin, and finally in the case of saccharose.

Experiments in the introduction of carbohydrates under soy and pea crops were carried out in order to ascertain their effect on the leguminous

plants, which are often sown under grain crops or, if sown alone, succeed the latter. The stubble of grain crops enriches the soil with carbohydrates in the form of cellulose, which is capable of intensifying the process of denitrification and, consequently, impoverishes the soil of mineral nitrogen. But when leguminous plants are sown with nodule bacteria present in the soil, cellulose ought to stimulate their growth.

The use of straw, as well as stubble, has a practical interest because of the fact that in a number of crops, straw causes a decreased yield. In this connection, the problem arises of the most productive utilization of straw as a fertilizer.

Quite significant findings on this question have been made by Thornton (1), Sabinin (2), Gray (3), Harper and Murrphy (4) and Ressel (5). An especially marked effect was shown by leguminous plants when using straw following PK and inoculating the soil with nodule bacteria.

We carried out an experiment in the use of straw under the pea and soy in greenhouse conditions in 1935, not at all for the purpose of verifiyng the work of the investigatons mentioned above, but in order to add to the factual material which might be used under agricultural conditions and to show how valuable the stubble of grain crops are for increasing yields.

Table 2

Serial No	Detail of the experiment	General yield (grain+straw)		Amount of nitrogen absorbed in milligrams		
		In grams per pot	In percentages	In above-ground parts	In roots	In total yield
	A. Without inoculation					
1	Control (without fertilizer) .	13.79	100.0	208.3	19.6	227.9
2	Soil + PK	16.25	122.9	216.9	17.1	234.0
3	» + PKN	33.76	241.9	—	—	—
4	» + 25 g of straw	13.24	95.2	150.1	—	—
5	» + PK + 25 g of straw .	16.77	111.1	272.0	—	—
6	» + 50 » » »	13.63	98.8	201.8	6.1	207.9
7	» + PK + 50 g of straw .	14.03	108.2	165.2	9.2	174.4
	B. Inoculation with nodule bacteria					
8	Soil (without fertilizer) . . .	19.29	139.8	545.9	39.1	585.0
9	» + PK	30.69	222.5	803.0	73.4	876.4
10	» + 25 g of straw	24.60	170.8	1 030.7	70.2	1 100.9
11	» + PK + 25 g of straw . .	33.11	247.3	1 542.9	71.4	1 614.3
12	» + 50 » » » . .	25.82	196.5	848.9	57.3	906.2
13	» + PK + 50 g of straw . .	34.14	254.8	1 031.0	72.5	1 103.5

Inasmuch as analogous results were obtained for soy and for pea in all experiments, we, for brevity's sake, present only the figures for soy. These demonstrate the following:

1) Chopped straw, under grain and legume crops, reacts negatively on the yield of soy and pea, owing partly to the denitrification it causes and partly to intensified reproduction of bacteria, which transform the mobile organic substances into a form difficult to assimilate.

2) When inoculated soil is fertilized with chopped straw, the yield of legumes is considerably increased, since there occurs an intensified growth of bacteria and assimilation of molecular nitrogen.

3) When chopped straw, inoculated with nodule bacteria, is used with PK, the yield of soy and peas is markedly increased, owing to intensified reproduction of bacteria in the presence of carbohydrates resulting from the decay of the straw.

4) The quantity and weight of nodules is markedly increased by the introduction of straw and bacteria, as likewise the content of nitrogen in the roots, seeds, straw and tubers.

5) The number of seeds in soy and pea is markedly increased in pots fertilized with PK and straw and inoculated with nodule bacteria.

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Received
7. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

GENETICS

INTERSPECIFIC HYBRIDS IN *SECALE* (RYE). I

SECALE CEREALE × SECALE ANCESTRALE, SECALE CEREALE × SECALE VAVILOVII, SECALE CEREALE × SECALE MONTANUM and SECALE ANCESTRALE × SECALE VAVILOVII HYBRIDS

By Dontcho KOSTOFF

(Communicated by N. I. Vavilov, Member of the Academy, 21. XII. 1936)

We have grown since 1931 (1), *Secale cereale* × *Secale montanum* hybrids and their progeny, while the other three hybrids we raised first in 1936. The cultivated species *Secale cereale* has unbrittle spikes while all the wild ryes have brittle spikes. F_1-hybrids have brittle spikes. The F_1-hybrids of the cross *cereale* × *montanum* has higly reduced fertility, while the others are highly fertile, especially the hybrid *S. cereale* × *S. ancestrale* showed an almost normal fertility. The hybrid *S. cereale* × *S. montanum* has about 75% viable pollen, while the other hybrids have a much larger percentage of viable pollen. The hybrid *Secale cereale* × *S. ancestrale* had almost normal pollen (93—97%). All hybrids were robust and produced numerous spikes.

Each *Secale* species used in our work had $n=7$ and $2n=14$ chromosomes (2). The F_1-hybrids *Secale cereale* × *S. montanum* had 5—7 bivalents, while all the other hybrids had usually 7 bivalents in the pollen mother cells during the first meiotic metaphase. The formation of one and very often of more than one chyasmata indicates that crossing over occurs between the chromosomes of the maternal and paternal species, which gives the possibility of recombination of characters between the cultivated and the wild ryes. The formation of seven bivalents in the hybrids studied, indicates that the genom of *Secale cereale* (S) is homologous with the genoms of *S. ancestrale*, *S. Vavilovii* and *S. montanum*.

The meiotic division in the hybrids: *Secale cereale* × *S. ancestrale*, *S. cereale* × *S. Vavilovii*, and *S. ancestrale* × *S. Vavilovii*, proceeds almost normally. Only slight irregularities were found, that lead occasionally to formation of non viable gametes. Meiotic division in *S. cereale* × *S. montanum*, however, had somewhat more irregularities. The occurrence occasionally of two univalents or an unsimultaneous separation of the bivalents during the first meiotic divisions were chiefly the causes for the irregularities observed. The occurrence of a relatively large percentage of abortive pollen in this hybrid seems to be partly due to the irregular meiosis and partly to the retar-

dation of certain processes proceeding in the anthers of the hybrid plants(1). The relatively high sterility of this hybrid is probably due not only to the relatively larger percentage of abortive pollen (than in the other hybrids) but to certain sterility factors of the type described by East in *Nicotiana*, which are also responsible for incompatibility of the cultivated rye when self pollinated.

The majority of the plants obtained from selfing *S. cereale*×*S. montanum*, and especially those produced from the back crosses of *S. cereale*× ×*S. montanum* to *S. cereale* are more fertile than the F_1-hybrids. In the F_4- and F_5-generation we obtained plants and lines very productive. Some plants of the F_2-generation were, however, completely sterile. We found even heteroploid forms in F_2-generation with 15 somatic chromosomes. One plant was found, that had 23 somatic chromosomes. These aberrants are unavoidable sequence from the irregularities in the reduction division of F_1-hybrids. The aberrant plants were completely sterile. The extra chromosome in the trisomics was normal in size, but it was not a fragment as in Gotoh's (1924, 1932) forms.

Fig. 1.

Fig. 2.

Fig. 1. Meiotic metaphase from a pollen mother cell of F_1 *Secale cereale*×*Secale ancestrale* hybrid with 7 bivalents.

Fig. 2. Meiotic metaphase from a pollen mother cell of F_1 *Secale cereale*×*Secale Vavilovii* hybrid with 7 bivalents.

Trisomic plants have probably originated after fusion of one gamete having 8 chromosomes with a normal game te having 7 chromosomes. Gametes with 8 chromosomes can be formed when 6 bivalents and 2 univalents are formed during the first meiosis if both univalents go to one of the poles (6+2=8).

The plant with 23 somatic chromosomes has probably orginated after fusion of one gamete having 16 chromosomes with a normal gamete having 7 chromosomes. Gametes with 16 chromosomes can originate from an 8 chromosomic nucleus formed after a chromosome distribution of 8—6, instead 7—7 during the first meiosis as just mentioned and then a doubling following non-occurrence of the second division.

Hybrids between cultivated rye and *S. montanum* have been occasionally produced by plant breeders (Duka). *S. montanum* is a perennial species and one can combine this character with the characters of the cultivated rye by hybridization. We wish to call the attention of the plant breeders that *S. ancestrale* might be used with success in plant breeding works when crossed with the cultivated rye. *S. ancestrale* is a productive and robust species and seems to be one of the most closely related species to the cultivated rye. *S. Vavilovii* behaves in the hybrids with cultivated rye in a similar way as *S. ancestrale*, but it is not so robust and so productive as *S. ancestrale*.

We are greatly indebted to Omelnitzkaja, A. Tichonova, N. Dogadkina and Z. Vishemirskaja for help throughout this work.

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Received
21. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS.
1937. Volume XIV, № 4

GENETICS

STUDIES ON POLYPLOID PLANTS

XVII. *NICOTIANA MULTIVALVIS* ($2n = 48$) × *NICOTIANA SUAVEOLENS* ($2n = 32$) AMPHIDIPLOID ($2n = 80$)

By Dontcho KOSTOFF

(Communicated by N. I. Vavilov, Member of the Academy, 21. XII. 1936)

Hybrids between *N. multivalvis* and *N. suaveolens* can be easily produced in both directions. They are self sterile and form usually abortive pollen. Only a very unsignificant percentage of the pollen stains deep red in aceto-carmine, which indicates that they probably are viable. In autumn, however,

Fig. 1. *A—Nicotiana multivalvis*; *B*—Amphidiploid, *N. multivalvis*×*N. suaveolens*; *C—N. suaveolens.*

when the temperature drops relatively low at night, the hybrids begin to form a somewhat larger amount of viable pollen. It seems that the low temperature influences the procedure of the meiotic processes inducing formation of restitution nuclei, dyads and pollen with somatic chromosome number. The observation on other hybrids (Kostoff, 1933) suggested that high temperature influence the procedure of the meiosis in a similar way and induce a formation of a larger amount of viable pollen with somatic chromosome number. Considering these circumstances, we attempted to increase the formation of viable non-reduce gametes under the influence of high tempe-

rature. Hybrids of the F_1-generation were put into thermostats with glass walls at a temperature of 40—42°C, 4 days for 3—4 hours each day. The buds and the flowers of the hybrids grown in more shady places of the greenhouse dropped down, while those of the hybrids grown near the opened windows at sunny places were not affected as much. When the small floral buds (in which it was supposed that meiosis occurs at the time of exposure at high temperature) developed into flowers, the plants were selfed and pollinated with *N. Sanderae* ($n=9$) having red flowers and then put again in the thermostat for four days, 3—4 hours each day. A large number of the flowers dropped. About 10 capsules did not drop at the time of exposure. Three reached the maturity. Twelve seeds were produced out of these three capsules. Nine of them were shrunken, one was very large but flat and two were round. Only one germinated and a plant developed that was very much like the F_1-hybrid *N. multivalvis*×*N. suaveolens*. This plant was self fertile. It usually had 40 chromosomes in the first and in each second metaphase.

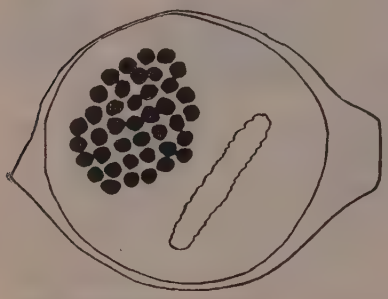

Fig. 2. Second metaphase of the amphidiploid hybrid *N. multivalvis*× *N. suaveolens* with 40 chromosomes (PMC).

These data show that the plant is an amphidiploid *N. multivalvis*×*N. suaveolens* plant. There are three most probable possibilities about the origins of this plant:

1. A non-reduced egg cell has been fertilized with a sperm having a non-reduced chromosome number. *N. Sanderae* pollen-tubes have made a path through the style of F_1-hybrid flower, thus facilitating the passing of the *multivalvis-suaveolens* pollen-tubes of pollen grains with somatic chromosome number through the style of the F_1-hybrid.

2. The plant has originated from an egg cell with 80 chromosomes formed following non-occurrence of both meiotic divisions *N. Sanderae* pollen tube has activated parthenogenetic development of this egg cell. Such cases occasionally occur in *Nicotiana* (species hybrids). Such a case was reported in *Triticum* too ([3]).

3. The plant has originated from an egg with somatic chromosome number and subsequent somatic doubling under the influence of the high temperature, the parthenogenetic development being induced by the pollen-tube of *N. Sanderae*.

The progeny of the amphidiploid *N. multivalvis*×*N. suaveolens* were uniform and like their parental amphidiploid plant and F_1-plants. The amphidiploids were more robust than F_1-hybrids and the flowers were slightly larger. They often formed flowers with more than five lobes and capsules with more than two carpels—characters that are typical for *N. multivalvis*. They were, however, less pronounced in the amphidiploids than in the maternal species. The seeds of the amphidiploids have traces of silver-like colour that is characteristic for *N. multivalvis*.

F_1-hybrids *N. multivalvis*×*suaveolens* have very irregular meiosis. They have usually only univalents or one bivalent during the first meiotic metaphase. Occasionally two bivalents were also observed, but pollen mother cells with two bivalents rarely occurred.

The reduction division in the amphidiploids is quite regular. We observed very rarely one polyvalent (trivalent) and one or two univalents. Lagg-

ing chromosomes on the spindle rarely occurred. The plants formed about 88 per cent viable pollen.

This high constancy can be explained by almost complete absence of bivalents in F_1-hybrids and polyvalents and univalents in the amphidiploids. The appearance usually of 40 bivalents in the first metaphase which originate from autosynthetic association between *multivalvis* homologous chromosomes as well as between *suaveolens* homologous chromosomes is an ultimate sequence from the absence of homology between *multivalvis* and *suaveolens* chromosomes (almost complete absence of allosyndesis).

It should be mentioned here that *N. multivalvis* is an american species while *N. suaveolens* is an australian species; hence the amphidiploid form combines the characters of an american and an australian species into a fertile plant.

An amphidiploid plant in crossing australian and american species was produced by Goodspeed (2). He produced it from *N. suaveolens* × *N. Bigelovii* hybrid and called it *Nioctiana disualovii* (1).

We are greatly indebted to Z. V. Vyšemirskaja, M. Linden and V. Bočarova for help throughout this work.

Institute of Genetics.
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Received
2. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PLANT PHYSIOLOGY

ON THE PHYSIOLOGICAL CHARACTERISTICS OF YAROVIZED AND NON-YAROVIZED PERILLA

By M. I. KLIUČNIKOVA

(Communicated by A. A. Richter, Member of the Academy, 7. XII. 1936)

While looking into the condition of field crops of perilla (*Perilla nankinensis*), we noted a very curious fact. Some plants of yarovized perilla brought from the field were always more withered than the non-yarovized. Naturally this fact whetted our interest. Preliminary experiments showed a difference in the general content of water and the field water deficit of the leaves of yarovized and non-yarovized perilla.

At the present time there exist a number of works which throw light on the physiological and biochemical peculiarities of yarovized plants and on their points of distinction from the non-yarovized.

In the sprouting seeds of winter wheats there take place, under the influence of yarovization, a reconstruction of the enzymatic apparatus as well as deep qualitative changes of the proteino-lipoid complex of the cell [Ricther (5)]. I. M. Vasiliev (1) notes a lowering of the frost resistance of winter wheats under yarovization. The direct influence of yarovization is manifested in a great accumulation of organic substances and an increased yield of seed (4).

The profound physico-chemical changes of the protoplasm of the embryo of yarovized seeds are transmitted to the adult plants. The investigations of I. A. Filippenko (7) have shown a different physical structure, a great chemical mobility of the protein substances of the leaves of yarovized winter wheats, a greater content of chlorophyll and a great activity of oxidizing ferments.

A study of the water regime of yarovized perilla plants has not only a theoretical but also a practical importance especially since in practice new questions crop up—questions of the drought resistance of yarovized plants of perilla and of the regions where it may be sown.

In the present work, we have investigated the water deficit in the leaves, the intensity of transpiration and the sucking force of the leaf cells of yarovized and control (non-yarovized) perilla plants, in so far as these data are important and determine the water regime of the plant.

The water regime of a plant depends both on environmental conditions and on the inherent properties of the plant itself. Inasmuch as the external conditions (soil moisture, meteorological conditions, etc.) were precisely

identical in our experiments, the differences found in the water regime of yarovized perilla must be attributed entirely to the indirect influence of the process of yarovization of the seed.

The variety of perilla used for experimentation was the Far Eastern, which reacts vigorously to yarovization. According to the findings of B. C. Zakharov, under field conditions in 1936, its vegetative period was shortened 11 days by yarovization. The seed was yarovized by Zakharov's method. Both jarovized seed and controls were sown at the same time in 6 rows, 50 plants to the row. In our own experiments the yarovized perilla flowered 5 days earlier than the control.

Three tests were taken during the course of the vegetativ period: namely, when 3, 6, and 8 pairs of leaves were well developed.

At the time of the third test, the control plants were in full bloom, while the yarovized had seeds in the milky stage.

Determinations of the water deficit of the leaves were made as follows. Completely healthy, uninjured leaves were clipped from the plant under water, weighed immediately and transferred for 30 min on water in order to restore turgor; the leaves were then taken out of the water, wiped with filter paper to remove excess of water droplets and immediately weighed. For determination of the rise of the water deficit, the leaves were left in the laboratory and weighed at definite intervals of time. On finishing the experiment they were dried to absolutely dry weight. The water content of the leaves was calculated in terms of percentages of the absolutely dry weight, while the water deficit was expressed in percentages of water content when saturation was complete.

From Table 1 it is evident that the absolute water content and the field water deficit in the leaves of yarovized perilla are greater than in the control. The intensity of the rise of water deficit and the range of variation of the water retention is also sharper in the former.

From a determination of the energy of the fall of water deficit, the following data were obtained.

The data of Table 2 show that the elimination of the water deficit under conditions of sufficient water supply is more intensive in the leaves of the

Table 1

Rise of Water Deficit in Perilla Leaves when Wilting. Samples taken at 8 o'clock in the morning

Nos of samples	Date of taking samples and number of whorl	Material	Percent of water	Water deficit			
				In the field	Duration of wilting (in hours)		
					2	4	23
1	21/VI	Control	854.6	7.1	8.6	41.6	68.6
	3	Yarovized	898.0	8.7	16.3	24.2	82.4
	3	Control	732.0	19.4	19.2	29.4	78.2
		Yarovized	802.9	21.9	25.3	33.5	86.6
2	19/VII	Control	492.0	20.0	21.2	28.2	73.0
	5	Yarovized	521.2	24.0	28.3	42.9	86.3
	3	Control	509.8	18.0	19.0	26.10	70.0
		Yarovized	617.0	20.4	28.6	38.7	82.17
3	23/IX	Control	264.2	9.2	16.0	26.3	62.9
	8	Yarovized	293.7	9.3	14.2	37.9	73.7

Table 2

The Fall of Water Deficit in Perilla Leaves under Conditions of Sufficient Water Supply

No of samples	Date of taking samples and number of whorl	Material	Water deficit in percentage			Restored on water	Condition of leaf
			Water deficit in field	When wilted after 20 hours	After restoration of the turgor on water		
1.	21.VII . . . . 2.5 p. m. 3	Control	22.2	64.2	44.7	19.5	Normal
		Yarovized	25.7	79.8	54.3	25.5	»
2.	23.IX . . . . 8 a. m. 8	Control	9.2	*62.9	40.4	22.5	»
		Yarovized	9.3	*73.7	45.3	28.4	»

yarovized perilla, which apparently suffers less from lack of water, notwithstanding its high water deficit during wilting.

The principal role in the determination of water retaining capacity belongs to the hydrophylous colloids and to osmotically active substances ([6]).

In our experiments we had to do with difference in the colloidal characteristics of leaves. Indeed, the values of the sucking force of leaf cells from the third whorl confirm this. In the case of leaves gathered at 9 o'clock in the morning, from plants having 6 pairs of well developed leaves, these were:

Control 0.7 Mol. of saccharose = 18.13 atmospheres
Yarovized 0.8 Mol. of saccharose = 20.90 atmospheres

In the case of plants left without roots to wilt for 20 hours in the laboratory, the respective figures were 0.1 and 1.3 Mol. of saccharose. The sucking force of the cells was determined by the weighing method. Out of a number of molar concentrations of saccharose, a solution was found such that the weight of an excised piece of leaf tissue (for all experiments the central part of the leaf blade, without the principale vein, was taken) floating on it, remained unchanged after 1, 2 and 20 hours.

The intensity of the transpiration of perilla leaves was determined under conditions of sufficient water supply. At 8 o'clock in the morning, when the plants had the least field water deficit as compared with the rest of the day, a few plants, as nearly alike as possible, having three or five pairs of well developed leaves, were taken up from the rows with the earth around them. Removal of the earth was effected in water, with the least possible loss of rootlets.

The plants were set in water in Erlenmeyer flasks.

To avoid evaporation, the mouth of the flask was covered with cotton carefully smeared with vaseline. A calculation by weight for the water evaporated after 6 hours was made. The determination for each plant was made 5 times. After the experiment, the leaves were cut off, traced on tissue paper, and their area thus calculated.

If the water supply will be sufficient (Table 3), the intensity of transpiration is higher in the yarovized perilla, and the difference in this respect from the control becomes sharper with the growth of the plant.

Not only do the yarovized plants have a greater energy of transpiration— they have a greater leaf surface. Possibly the latter fact explains the greater

* Wilted after 23 hours.

Table 3

Intensity of Transpiration of Perilla Leaves

No of samples	Time of taking samples	Material	Intensity of transpiration in grams per each dm^2 of leaf area	Leaf surface per each gram of absolutely dry weight	Ratio of absolutely dry roots to absolutely dry weight of overground organs
1	25.VI	Contol	100%	100	—
		yarovized	107.4%	115.9	—
2	10.VII	Control	100%	100	11:11.74
		yarovized	119.4%	142.3	1:12.9

accumulation of dry substance in the overground parts of the yarovized perilla.

The increase of transpiration in yarovized perilla may depend on the anatomical structure of the leaf, as well as on the physico-chemical properties of the plant. According to V. P. Zalenskij (3), a great number of cells per unit area and a large content of chloroplasts in the case of xeromorphous structure lead to a great absorption of solar energy and, consequently, to an intense transpiration. According to E. A. Žemchužnikov (2), the intensity of transpiration may depend on the degree of openness of the stomata. The leaves of the perilla gave a stomatal apparatus on both the upper and the lower sides of the leaf. On the leaves of the third whorl, both in the control and in the yarovized plants, one and the same ratio was found between the lower stomata and the upper ones namely, 24:5. Progressively, in the higher whorls, the size of the epidermis cells of the leaves decreases, while venation increases. The number of stomata in the field of vision increases, but the difference in amount of openness of the stomata and the number of stomata contained in a unit area could not be determined.

Conclusions. The water regime in the leaves of yarovized perilla presents the following differences from that of the control: 1) a greater content of water in the leaves; 2) a greater field water deficit, and a more intense growth of the latter under conditions of insufficient water supply; 3) a greater transpiration energy under conditions of sufficient water supply; 4) the sucking force of the cells of the leaf tissue is greater in the yarovized plants.

A difference between control and yarovized plants in the number of stomata per unit area and in amount of openness of the stomata could not be established.

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Received
7. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSIOLOGIE VÉGÉTALE

INFLUENCE DES DIFFÉRENTES SUBSTANCES ARTIFICIELLEMENT INTRODUITES DANS LES FEUILLES DU TABAC SUR LE DÉVELOPPEMENT DE LA MALADIE APPELÉE «RIABOUKHA»

Par N. I. EFIMOVA

(Présenté par V. N. Lubimenko, de l'Académie Ukranienne, le 7. XII 1936)

La fréquence des taches maladives observées sur les feuilles du tabac, ayant pour résultat un abaissement considérable dans la quantité et la qualité de la récolte, a attiré l'attention sur l'étude de cette maladie et des moyens propres à la combattre. Un des premiers essais de ce genre a été le travail de D. Ivanovsky et V. Polovtsev, publié en 1890 (1).

Il faut reconnaître cependant que ces auteurs n'ont pu découvrir les vraies causes de la maladie et ce n'est qu'en 1917 que Wolf et Foster (2) sont arrivés à dégager le microbe (*Bacterium tabacum*), provocant cette maladie, ce qui a attiré définitivement l'attention à cet ennemi de la culture du tabac.

Plusieurs auteurs ont ensuite confirmé le caractère bactérial de cette maladie du tabac, et Kokhanovskaja (3) l'a démontré pour la makhorka et a étudié les voies possibles de la contagion des feuilles.

Malgré l'apparition d'une série de travaux bactériologiques postérieurs, l'analyse physiologique ne s'est presque pas portée jusqu'à présent sur l'influence des conditions du milieu et de l'état de la plante sur le développement de la maladie. On ne trouve dans la bibliographie que des indications très générales sur l'influence des conditions méthéorologiques sur la propagation et le développement de la maladie chez la makhorka (4), ainsi que sur le rôle de l'engrais (5).

C'est pourquoi nous avons décidé d'élucider l'influence des conditions de nutrition de la plante sur la susceptibilité de contagion. En premier lieu, nous avons essayé de faire des expériences de laboratoire pour l'influence de l'introduction artificielle de certaines substances minérales et organiques sur le développement de la maladie.

Deux sortes de tabacs étaient prises pour ces expériences: le Trapésonde et la Platana Arkadia. Les graines des deux sortes, obtenues en lignées pures, ont été reçues du département de sélection de l'Institut de tabac et de makhorka (à Krasnodar).

Au commencement de la floraison, des feuilles tout à fait identiquement développées du 6-me et du 7-me étage ont été coupées de la plante sous l'eau. Ces feuilles ont été ensuite artificiellement infectées par la culture du mi-

crobe à l'aide de piqûres, faites par une aiguille. Cette infection n'était importée que dans la partie gauche de la feuille, à partir de la nervure médiane, tandis que la partie droite subissait la même quantité de piqûres, mais faites par une aiguille stérile.

L'infection était produite par une culture de bouillon de trois jours de *Bact. tabacum*, dégagée par N. S. Novikova et diluée par de l'eau stérilisée.

Tout de suite après l'infection les feuilles étaient placées dans des verres, contenant des solutions de différentes substances, ainsi que sur de l'eau distillée, d'après le schéma suivant.

1. Eau distillée
2. Solution de KH_2PO_4 . . . . . 0.1%
3. » » $Ca(NO_3)_2$ 0.1%
4. » » KNO_3 . . . . . 0.1%
5. » » saccharose 10%
6. » » » 5%

Trois séries d'expériences ont été faites, répétées chacune deux fois. Immédiatement après le transport des feuilles infectées sur les solutions indiquées, les grands verres, contenant les feuilles, étaient placés dans des thermostats. Les expériences ont été faites toujours à deux températures: dans un thermostat la température était de 27—28° C, et dans l'autre, de 14—15° C. Les feuilles restaient sur les solutions pendant 8 jours et chaque jour on faisait des observations sur le développement de la maladie.

Résultats des expériences

Les observations faites ont démontré que sur toutes les solutions employées pour les deux espèces de tabac, le developpement de la maladie se manifestait avec beaucoup plus d'intensité à la température de 27—28° C qu'à celle de 14°—15° C.

En ce qui concerne les différentes substances employées, nous avons trouvé que sur le saccharose de la concentration de 10% les feuilles noircissent et se fanent prématurément. C'est pourquoi les feuilles placées sur cette solution, ont été ensuite éliminées de l'expérience.

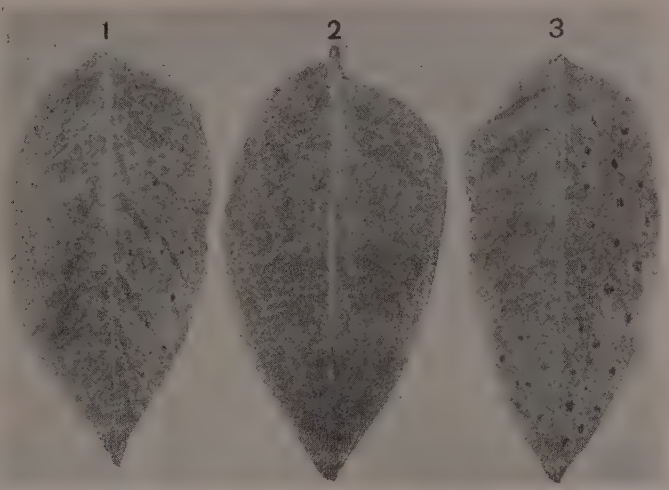
Pour les autres solutions nous avons remarqué, que le développement le plus intense de la maladie avait lieu dans les feuilles, placées sur les solutions de $Ca(NO_3)_2$; un peu plus faible, sur les solutions de KH_2PO_4 et KNO_3; encore plus faible, sur l'eau distillée, et tout à fait insignifiant, sur la solution de 5% de saccharose.

Des différences très marquées ont été observées pour les feuilles placées sur les différentes solutions, au point de vue de l'état général et du caractère des taches de la maladie développée. Ainsi, sur les solutions de KH_2PO_4 et de KNO_3 on a toujours obtenu sur les feuilles de grandes taches sombres, entourées d'une large auréole jaune; sur la solution de $Ca(NO_3)_2$ elles étaient encore plus grandes et leur partie centrale montrait après 5 jours d'infection une nécrose complète du tissu. Au contraire, sur la solution de 5% du saccharose on n'observait qu'un léger éclaircissement aux endroits de piqûre, aucune tache sombre ne s'étant formée.

L'aspect général des plantes infectées de Platana Arkadia, obtenu après une infection de 5 jours, est représenté sur la photographie. On voit ici les feuilles: 1) sur l'eau distillée, 2) sur la solution de saccharose de 5% et 3) sur la solution de 0.1% de $Ca(NO_3)_2$. Le plus grand développement des taches peut être observé pour la feuille sur la solution $Ca(NO_3)_2$, un développement beaucoup plus faible, pour celle sur l'eau distillée; chez les feuilles

posées sur la solution de 5% de saccharose, les taches manquent complètement. Quant aux feuilles qui se trouvaient sur la solution de $Ca(NO_3)_2$, les taches s'y sont développées à la fin de l'expérience même sur la partie droite des feuilles, qui n'avaient pas été infectées (piqûres faites par l'aiguille stérile).

En ce qui concerne la différence des sortes, il est à remarquer que les taches se sont développées plus fortement sur les feuilles du Trapésonde que sur celles de Platana Arkadia sur toutes les solutions, y compris la solution



Feuilles du tabac Platana Arkadia cinq jours après l'infection. Dans chaque feuille la partie gauche a été infectée: *1*—feuille sur l'eau distillée: *2*—sur la solution de saccharose de 5%; *3*—sur la solution de nitrate de chaux de 0.1%.

Knop employée à titre de comparaison dans la deuxième et troisième série d'expériences.

Nous considérons ces expériences comme préliminaires. Nous les avons faites préalablement pour nous orienter dans les études plus approfondies de l'influence de différents sels sur la susceptibilité à l'infection de differentes plantes de tabac.

Cependant les résultats obtenus ont donné certaines indications, concernant le rôle des différentes substances dans le développement de la maladie.

Dans le cas de manque de substances hydrocarbonées, les feuilles des plantes de tabac (restées dans l'obscurité) se montraient assez susceptibles à l'infection. L'introduction de différents sels [KN_2PO_4, KNO_3. $Ca(NO_3)_2$] surtout l'introduction de la solution $Ca(NO_3)_2$ dans les feuilles des deux espèces de tabac expérimentées accélère incontestablement la maladie. Ce fait se rattache aux données des analyses de Vityne, qui a trouvé dans les feuilles infectées par le *Bact. tabacum* 5.14% de calcium, tandis que les feuilles non infectées n'en contenaient que 3.78%.

L'introduction du saccharose dans la feuille retarde sensiblement l'apparition de la maladie. La cause en peut être attribuée ou bien à la réaction spécifique de la maladie vis-a-vis de la concentration du sucre dans le

substratum nutritif (dans les cellules de la feuille), ou bien à l'augmentation de la pression osmotique; les expériences futures montreront, à laquelle des deux causes le retard doit être attribué. Cependant elles peuvent agir ensemble.

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Manuscrit reçu
le 7. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PHYSIOLOGIE VÉGÉTALE

EXPÉRIENCES SUR L'ÉCLAIREMENT DES GRAINES PAR LA LUMIÈRE DE DIFFÉRENTE LONGUEUR D'ONDES

Par A. A. KUZMENKO

(Présenté par V. N. Lubimenko, de l'Académie Ukranienne, le 7. XII. 1936)

L'étude du développement de l'organisme végétal, ainsi que la recherche de méthodes permettant de régler ce développement ont attiré l'attention des savants de l'URSS. Dans les dernières années un intérêt particulier a été éveillé pour l'étude de l'influence de l'énérgie rayonnate sur les organismes végétaux. Cependant l'action des rayons des différentes parties du spectre visuel produite pendant les premiers stades de la croissance des plantes et son influence sur leur développement postérieur a été seulement très peu étudiée. Suivant l'initiative de V. Lubimenko, nous avons entrepris cette année des expériences avec quelques plantes pour élucider cette question.

Nous avons pris pour ces expériences des graines de sortes pures de deux espèces de blé (*Tr. vulgare.* var. *lutescens* 062 et *Tr. durum* var. *melanopus* 069) de sudza (*Perilla ocymoides* sorte d'Amour) et de plusieurs espèces de makhorka* et de tabac. Les graines de ces plantes étaient exposées pendant leur germination à l'influence de la lumière de différente couleur ou de l'obscurité pendant 10 à 15 jours. Ensuite les jeunes plantules étaient transplantées dans des vases de culture où elle croissaient dans des conditions naturelles d'éclairement.

Une installation spéciale avec une source constante de forte lumière électrique fut établie pour étudier les graines en voie de germination. L'expérience exigeait que les graines fussent simultanément exposées à l'action de l'obscurite, de l'éclairement électrique continu (lumière blanche), de la lumière jaune-rouge et de la lumière bleue, dans des conditions identiques de température et d'humidité. Pour obtenir la lumière colorée nous nous sommes servis de filtres liquides: solution de 5% de bichromate de potasse et solution ammoniacale de 15 a 20% d'oxyde de cuivre, renfermées dans des vases de verre clos. L'épaisseur de la couche du liquide était choisie et vérifiée spectroscopiquement de manière que l'écran contenant le premier liquide dégage la partie du spectre de λ 750 à 575 mμ et celui avec le second—la partie de λ 525 à 390 mλ. La variante éclairée par la lumière blanche avait comme écran la couche de 5 cm d'eau distillée.

* *Nicotiana rustica.*

Comme sources de lumière on a employé des lampes de 300 et de 500 W placées à la distance de 25 cm du filtre. Les graines étudiées, mises à la distance de 20 cm des filtres et de 45 cm de la lampe, étaient disposées toujours en une seule couche dans des petits verres blancs fermés.

La température était soutenue à 28°C environ, humidité étant limitée pour empêcher la croissance rapide des plantules. Au moment de la transplantation des plantes dans les vases, la longueur totale des plantules ne surpassait pas 5 cm pour le blé, 1.5—2 cm pour le soudza et 1 cm pour le tabac.

Blé. Les graines germées du blé mou 062 et du blé dur 069 restaient sous l'action de la lumière électrique ordinaire (blanche), la lumière rouge-jaune et la lumière verte-bleue pendant 10 jours (I série) et 15 jours (II série). Ensuite elles étaient plantées dans des vases de culture contenant de la terre du type de tschernozem, où elles croissaient jusqu'à l'époque de la maturité. Les observations, concernant certains résultats de l'expérience sont données dans le tableau 1 et sur les figures 1 et 2.

Fig. 1. Blé mou 062 des graines éclairées pendant 10 jours par: *1*—la lumière blanche; *2*—la lumière rouge-jaune; *3*—la lumière verte-bleue.

Fig. 2. Blé dur 069. Les mêmes notations que sur la fig. 1.

Outre l'accélération du développement des plantes exposées à l'influence de la lumière rouge-jaune on a constaté dans les deux expériences avec le blé mou une augmentation de la hauteur et l'accroissement de la masse totale des plantes en comparaison avec la variante éclairée par la lumière blanche. Cette réaction est plus fortement exprimée pour le blé mou 062 que pour le blé dur 069. Chez les plantes de cette dernière espèce on a pu observer dans la vitesse de leur développement des différences peu marquées entre les variantes éclairées par différentes lumières.

Tableau 1

Sorte de blé	Qualité de la lumière	I série des plantes	II série des plantes				
			Dates				
		Date de la formation de l'épi	d'ensemencement	de la formation de l'épi	Nombre d'épis (25/VII, 5 plantes)	Poids de graines en g (5 plantes)	Hauteur des plantes en cm
062	1. Blanche	6/VII	26/V	10—VII	4	2.76	34
	2. Rouge-jaune	1/VII	26/V	7—VII	8	3.40	48
	3. Bleue	2/VII	26/V	13—VII	3	3.70	48
	4. Obscurité	—	26/V	14—VII	9	4.10	50
069	1. Blanche	4/VII	26/V	11—VII	8	—	45
	2. Rouge-jaune	2/VII	26/V	9—VII	5	—	59
	3. Bleue	3/VII	26/V	10—VII	6	—	56
	4. Obscurité	—	26/V	15—VII	6	—	49

Perilla ocymoides. Les graines germées du soudza, éclairées pendant 10 jours, étaient plantées ensuite dans un vase de culture, où elles croissaient jusqu'à l'époque de la maturité des graines. Les résultats sont donnés dans le tableau 2.

Tableau 2

Qualité de la lumière	Date du commencement de la floraison	Poids moyen de la substance sèche d'une plante en g	Poids des graines d'une plante en g
1. Blanche	15.VIII	6.18	1.64
2. Rouge-jaune	9.VIII	6.23	2.13
3. Bleue	12.VIII	10.08	2.27

Ces données montrent que l'action de la lumière rouge-jaune sur les graines germées du soudza, ayant duré 10 jours, accélère l'époque de la floraison de 6 jours par rapport à la variante des graines, placées dans les mêmes conditions de température et d'éclairement continue par la lumière blanche.

Tabac. Les graines germées de différentes espèces de tabac et de makhorka étaient éclairées pendant 10—15 jours et plantées ensuites dans des vases du culture. Pendant la première période on a pu observer une accélération marquée de la croissance chez les plantes développées des graines éclairées par la lumière rouge-jaune et un ralentissement pour celles éclairées par la lumière bleue. Au stade du développement postérieur ces différences se perdaient pour certaines espèces, et se conservaient pour d'autres.

Comme on peut voir sur la fig. 3, cette différence était très marquée pour l'espèce Varatik, non seulement au point de vue du moment de commencement de la floraison, mais aussi du nombre des feuilles et de leur surface.

Fig. 3. Tabac Varatik des graines éclairées par: *1*—la lumière blanche: *2*—la lumière rouge-jaune; *3*—la lumière verte-bleue.

Les analyses chimiques, que nous avons faites pour les feuilles de certaines espèces recueillies, prouvent que le traitement auquel nous avons soumi les graines influe aussi sur la quantité de nicotine dans la feuille.

Les données obtenues montrent l'effet favorable qu'on peut obtenir par l'emploi des méthodes étudiées pour l'accélération et l'augmentation de la récolte de certaines plantes. Le travail est continu, appliqué à de nouvelles plantes et de nouvelles variantes d'étude.

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Manuscrit reçu
le 7. XII. 1936.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PLANT PHYSIOLOGY

THE INFLUENCE OF NODULE BACTERIA AND THE AZOTOBACTER ON THE YIELD OF LEGUMINOUS AND CEREAL PLANTS. SOWN TOGETHER

By T. T. DEMIDENKO and E. F. TIMOFEIEVA

(Communicated by A. A. Richter, Member of the Academy, 13. XII. 1936)

In order to obtain the best yields from cultivated plants it is necessary to know all their peculiarities and the most favourable conditions for their development.

The way in which they are sown exerts a great influence on the development of plants. Most crops require pure sowings, but some of the legume crops, especially those used forage, develop better when sown together with other crops.

Because of the weak development in the mechanical tissues of their stalks, the vetch and pea, when sown alone, are liable to lie flat on the ground, but when sown together with cereals, such as the oats for example, which furnishes a support around which the weakstalked vetch can twine or lift itself by means of tendrils and so maintain a vertical position, the legumes can better avail themselves of sunlight and carbon dioxide.

A most interesting phenomen was observed in the experiments of Kosovich, Domontovich and Polosin ([5],[6],[7]), namely: when the cereals were sown together with plants characterized by acid root excretions that contributed to the disintegration of phosphorite and its conversion from a difficultly soluble form to one easily assimilable, they develop well, despite the fact that they could not to a sufficient extent directly utilize the phosphorite the first year.

Now if there exists between plants disintegrating the difficultly assimilable sources of mineral nutrition, but unable to use it, a relation, when sown together, under which one plant sets up favourable conditions for the other, then it would seem quite promising to study the mutual relations between leguminous and cereals, sown together, with an inoculation of the soil with nodule bacteria and the azotobacter.

The leguminous feed on atmospheric nitrogen, through the agency of nodule bacteria, while certain plants which accumulate considerable amounts of sugars in their roots enable the azotobacter to form a rhizosphere and to develop vigorously. By fixing atmospheric nitrogen the azotobacter converts it into ammonia and produces an assimilable form of nitrogen for the higher

plants, according to investigations of Richter, Kostytschev, Sheloumova and Vinogradsky ([1], [2], [3], [4]).

By sowing leguminous with plants that store up carbohydrates in their roots and by inoculating the crops with nodule bacteria and the azotobacter, both separately and together, we hoped to learn: first, to what degree the nitrogen fixed by nodule bacteria was available to the grain plants; second, whether the leguminous could utilize the nitrogen fixed by the azotobacter, and third the role of the nitrogen obtained from the nodule bacteria and azotobacter in the nutrition of the higher plants.

The results and the details of the experiments are shown in the table

Table

Influence of the Nodule Bacteria and Azotobacter on the Yield of Oats + Peas

Serial No	Details of the Experiment	Number of plants per pot	Weight of oat grain		Total yield of oats (grain + straw)		Weight of pea grain		Total yield of pea (grain + straw)	
			Grams	%	Grams	%	Grams	%	Grams	%
1	Oats (soil without fertilizer)	12	7.29	100	17.47	100.	—	—	—	—
2	Oats + azotobacter .	12	8.66	118.7	19.79	113.2	—	—	—	—
3	Peas (soil without fertilizer)	6	—	—	—	—	13.8	100	31.52	100
4	Peas + nodule bacteria	6	—	—	—	—	16.29	117.9	40.36	128
5	Oats + peas	12+6	7.03	96.4	16.55	95.3	2.27	16.4	7.52	23.8
6	Oats + peas + nodule bacteria	12+6	11.33	155.4	25.50	145.9	3.83	27.7	12.18	38.6
7	Oats + peas + azotobacter	12+6	14.32	196.5	29.84	170.8	3.87	28.0	13.59	43.1
8	Oats + peas + nodule bacteria + azotobacter	12+6	14.40	197.5	33.22	190.1	4.11	29.7	15.66	49.6

From these data, the following is evident: the yield of both grain and straw is increased in oats by inoculation of the soil with the azotobacter; in the same way, the pea reacts positively to inoculation with nodule bacteria. When oats and peas are sown together, but without inoculation, the oats grow better than the peas, which are somewhat cramped in a mixed sowing. This may be explained by the fact that oats more energetically absorb the nutrient substances than peas, so that, the supply being limited, the peas have to get along with the nitrogen salts that the oats leave them.

In mixed sowings of oats and peas with inoculation of the soil with nodule bacteria, there is a marked fall in the yield of peas and an increased yield of oats, which would seem to have partly intercepted the nitrogen fixed by the nodule bacteria.

In this case, the result was just the same as in Virtanen's experiments([8]) with mixed sowings of soy and oats under sterile conditions and inoculation of the nutrient medium with a pure culture of nodule bacteria: he also observed a better growth of the oats. But there arises the question of why it is so: Is it that the nodule bacteria discharge part of the ammonia or some

other nitrogen compound into the surrounding solution on which the oats feed? Is it that the nodule bacteria discharge a ferment that can fix nitrogen? Or can it be conjectured that the nodules of the pea have the property of quickly disintegrating and freeing the ammonia, which is immediately absorbed by the roots of the plant? Or, finally, as is hard to suppose, is there an exosmosis in the pea, which takes place in leguminous according to Virtanens data, with discharge of amino-acids from the roots into the soil solution as a result of slight permeability of the cell membranes for substances of high molecular weight?

When the soil in pots containing mixed sowings of oats and peas is inoculated with the azotobacter, a favourable influence of the azotobacter is observed on the growth and yield of oats, which, it would appear, supply energizing material to the bacteria, which assimilate molecular nitrogen, the latter in its turn being partly used for feeding the oats.

The pea, when grown together with oats without nodule bacteria, developed fairly well, owing, it would seem, to the fact that it had at its disposal a certain quantity of nitrogen that it had accumulated from the mineralization of organic substances in the soil, but which was not absorbed by the oats.

When mixed sowings of oats and peas were grown on soil inoculated with nodule bacteria and the azotobacter, each of the plants developed quite well, since the leguminous were able to obtain a sufficient quantity of nitrogen due to the activity of the nodule bacteria, while the oats drew, in part on the peas, in part, on the azotobacter.

On the basis of the data obtained from mixed sowings, the following conclusions may be drawn:

1) The nodule bacteria supply nitrogen not only to the peas on whose roots they grow, but also to oats that grow in the same soil.

2) In this connection, it may be supposed that the assimilation of nitrogen by oats is due to the ammonia discharged by the leguminous in considerable quantity during the development of nodule bacteria. Probably they fix so much nitrogen that the plant cannot absorb the whole amount of the ammonia and discharges it into the surrounding solution, and this enables the oats to use it for synthetic purposes.

3) In mixed sowing of oats and peas with inoculation of the soil with nodule bacteria, the peas contribute to the development of the oats, yielding up to them the nitrogen assimilated by the bacteria.

4) When oats and peas are sown together and the soil is inoculated with a mixed culture of azotobacter and nodule bacteria, the influence of such sowing is favourable for the oats and somewhat depressing on the peas, at the expense of which, to a certain extent, the oats develop.

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Received
13. XII. 1936.

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Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XIV, № 4

PLANT PHYSIOLOGY

ON THE CAUSES OF INTERMEDIATE COLD-RESISTANCE IN WHEAT HYBRIDS OF THE FIRST GENERATION

By M. I. SALTYKOVSKIJ

(Communicated by A. A. Richter, Member of the Academy, 13. XII. 1936)

Analyses of the centre of injury in the tissues of winter wheats conducted in the laboratory of A. A. Richter justify the conclusion that the older the tissues and individual cells within one and the same tissue, and the more they are differentiated, the less they are hardened to low temperatures ([1]).

On this basis, we studied the influence of the age of the plant on its cold-resistance. Field and laboratory experiments with various summer and fall dates of sowing rye and wheat (1925/1926, 1926/1927 and subsequent years) showed that very early sowings, giving «older» plants, are less cold-resistant than later sowings ([2,3]).

Later on, with the appearance of T. D. Lysenko's theory of the stage development of plants, there began broad investigations of the significance of development for hardiness of winter cereals. In no small degree these investigations were faciliteted by the fact that the author proposed, at the same time he enunciated his theory of stage development, a very simple and convenient method of controlling the development of plants, the method which was named «yarovization».

It was thus made possible to divide the development of the plants into different stages, in passing through which, plants need different conditions, specific for the different and definite stages of development ([4,5]) .

Of recent years, the works of a number of investigators have established the fact that fall sowings of winter grains with yarovized seed possess very little resistance to cold ([7,8,9,11]). In explanation of this fact, a number of investigators consider that with the finishing of the stage of yarovization, the cold-resistance of winter cereals falls sharply and that there is no capability in yarovized plants of being hardened to low temperatures when they have passed the stage of yarovization ([7]).

Our work showed the incorrectness of such a conclusion. The plants retain their capability of being hardened do low temperatures even after passing through the stage of yarovization, their resistance to cold can greatly increase after the passage through the stage of yarovization. Hardening and the capability of being hardened fall markedly only with development into the light stage ([9, 10, 11]).

On the basis of these views concerning the progressive loss in wheat plants of the ability to become hardened to cold concomitantly with their development into the light stage, we will analyse the causes of intermediate inheritance of cold-resistance in wheat hybrids (12).

The experimental material for the present paper was obtained at the Saratov Plant Breeding Station in 1929, 1930. The experiments were started with the purpose of studying the connection between cold-resistance and duration of the vegetative period of wheats, on which, as we then thought, must depend, in great degree, the age at which they go into wintering.

A few words about the material and the method of the experiments. The pairs taken for crossing had, in respect to stages of development, the following general characteristics, Winter wheats 0329, A2453 and 0121 have long stages of yarovization and light stages. The light stage in 0121 is somewhat shorter than in the two others. In winter wheat Kooperatorka, both stages are relatively long but shorter than in the foregoing group of varieties. All other wheats were spring wheats. Their stage of yarovization is short. With respect to the light stage, these wheats constitute a series from a very long stage to an ultra-short one in the following order: 0321, Marquis, 0841, 0721, 062, 2184, Prelude. In the three medium spring wheats, the light stage is approximately the same.

Tests of the cold-resistance of the hybrids and the parent forms were carried out in the field. The details of the experiment have been described in a former paper, hence we will not repeat them here (12). Two spring sowings were made in pots with seed yarovized at 0° for 67, 57, 46, 36, 24 14 and 4 days. Control pots were sown with non-yarovized seed. The control seed before sowing, as the yarovized seed before yarovization, was soaked for 24 hours. The seed was sown in the greenhouse on April 12, 1930. After tillering had begun, all pots were transferred to the open air. All observations were carried out on each plant separately. In the tables are shown the dates of earing for the majority of the plants of both sowings.

Table 1

Frost Resistance and Earliness of F_1-Hybrids and Their Parent Forms

Combinations	Dates of earing of sowings made on April 12 1930			Percentages of sowings made on September 5 1929, 1930 that survived the winter		
	Parent forms		F_1-Hybrids	Parent forms		F_1-Hybrids
	Winter wheats yarovized ♀	Spring wheats non-yarovized ♂		Winter wheats yarovized ♀	Spring wheats non-yarovized ♂	
Kooperatorka ♀ × Prelude ♂	14 VI	29 V	8 VI	96	0	0
» ♀ × 2184 ♂	14 VI	3 VI	8 VI	96	0	6
» ♀ × 0721 ♂	14 VI	8 VI	14 VI	96	1	76
» ♀ × 062 ♂	14 VI	10 VI	13 VI	96	1	85
» ♀ × 0321 ♂	14 VI	27 VI	5 VII	96	95	96
0329 ♀ × Prelude ♂	25 VI	29 V	12 VI	100	0	50
0329 ♀ × 2184 ♂	25 VI	3 VI	12 VI	100	2	86
0329 ♀ × 0721 ♂	25 VI	8 VI	17 VI	100	4	95
0329 ♀ × 062 ♂	25 VI	10 VI	17 VI	100	3	95
0329 ♀ × Marquis ♂	25 VI	13 VI	21 VI	100	10	98

Table 2

Dates of Earing of Hybrids and Parent Form after Yarovization.
Experiments carried on in Saratov, in 1930

Groups of hybrids	Combinations and parent forms	Control	Duration of yarovization, in days						
			4	14	24	36	46	57	67
1	Kooperatorka	—	—	—	27 VII	11 VII	31 VI	14 VI	15 VI
	Kooperatorka×0329	—	—	—	11 VIII	8 VIII	6 VII	22 VI	15 VI
	0329	—	—	—	—	—	16 VII	29 VI	25 VI
	0121×Kooperatorka	—	—	—	10 VIII	27 VII	16 VII	29 VI	16 VI
	0121	—	—	—	—	25 VIII	21 VII	25 VI	18 VI
	0329×A2453	—	—	—	—	—	14 VII	28 VI	26 VI
	A2453	—	—	—	—	25 VIII	7 VII	24 VI	26 VI
2	Kooperatorka×0321	5 VII	5 VII	6 VII	5 VII	4 VII	1 VII	20 VI	16 VI
	0321	27 VI	29 VI	28 VI	27 VI	25 VI	25 VI	25 VI	24 VI
	Kooperatorka×0841	21 VI	15 VI	14 VI	12 VI	13 VI	12 VI	8 VI	8 VI
	0329×0841	22 VI	18 VI	18 VI	15 VI	16 VI	13 VI	12 VI	10 VI
	0841	8 VI	8 VI	8 VI	8 VI	8 VI	10 VI	10 VI	10 VI
	0329×Marquis	21 VI	20 VI	21 VI	21 VI	18 VI	18 VI	16 VI	12 VI
	Marquis	13 VI	12 VI	12 VI	12 VI	13 VI	12 VI	12 VI	13 VI
3	Kooperatorka×Prelude	8 VI	8 VI	5 VI	5 VI	3 VI	3 VI	3 VI	3 VI
	0329×Prelude	12 VI	12 VI	12 VI	15 VI	12 VI	13 VI	12 VI	0
	Prelude	29 V	29 V	30 V	29 V	31 V	31 V	31 V	3 VI
	Kooperatorka×0721	14 VI	15 VI	14 VI	14 VI	13 VI	13 VI	14 VI	15 VI
	0329×0721	17 VI	16 VI	17 VI	15 VI	15 VI	15 VI	15 VI	17 VI
	A2453×0.21	20 VI	17 VI	17 VI	16 VI	17 VI	17 VI	16 VI	15 VI
	0721	8 VI	8 VI	0	8 VI	10 VI	13 VI	10 VI	10 VI
	Kooperatorka×062	13 VI	13 VI	15 VI	16 VI	16 VI	14 VI	16 VI	0
	062×0329	17 VI	14 VI	15 VI	17 VI	15 VI	16 VI	16 VI	15 VI
	062	10 VI	8 VI	10 VI	10 VI	12 VI	12 VI	12 VI	0
	Kooperatorka×2184	8 VI	4 VI	8 VI	8 VI	8 VI			
	329×2184	12 VI	8 VI	12 VI	15 VI	15 VI			
	2184	3 VI	3 VI	3 VI	3 VI	2 VI			

Note. In the above table, a dash signifies that in the corresponding variant, earing did not take place until the late fall. A zero indicates an absence of data.

Let us review briefly the experimental material. Two sets of data are given in the first table: the earliness of spring wheats, of hybrids from the crossing of spring wheats with winter wheats, and of yarovized winter wheats, on the one hand, and on the other, the cold-resistance of analogous material in field sowings in 1929, 1930. It is sufficient merely to glance at the table to see the close correlation between the earliness of spring wheats and hybrids and their resistance to cold. The later the variety or hybrid when sown in the spring, the higher its cold-resistance when sown in the fall, and conversely.

In our experiments of 1930 all F_1 hybrids eared later than the parent spring form (Tables 1 and 2), and their cold-resistance was invariably higher than that of the latter.

The results of the experiment in yarovization of hybrids carried out in 1930 are shown in Table 2 and may be roughly divided into three groups.

First Group. These are hybrids of winter wheat×winter wheat. After complete yarovization, F_1 is equal in earliness to the earlier of the parent forms, that is to the parent having the shorter light stage. In sowings with incompletely yarovized seed F_1, in all the combinations and variants of this group, occupies an intermediate position between both parents in the matter of earliness. For example, Kooperatorka with 67-day yarovization eared on June 15. Its hybrids with the 0329 and 0121 lines of wheat, with 67-day yarovization, eared on June 15—16. In the variant with 57-day yarovization, Kooperatorka eared on June 14, while its hybrids with the 329 and 121 pure lines eared on June 22 and June 29 respectively, and so on.

Second Group. F_1-hybrids of winter wheat×spring wheat whose development was greatly hastened by yarovization. Such combinations fall in this group as Kooperatorka×0321, Kooperatorka×0841, 0329×0841, and 0329×Marquis. The hastened development resulting from yarovizing the hybrids of these combinations was, respectively, 19, 13, 12 and 9 days. Just as in the first group, under prolonged yarovization the dates of the hybrids earing coincide with the earing dates of the parents whose light stages were the shorter. We will cite the most graphic example, that of F_1-hybrids of Kooperatorka×0321. Without yarovization, the latter eared 8 days later than the spring wheat parent of pure line 0321, while after yarovization the opposite is to be observed; that is, the hybrids eared 7 days earlier than spring wheat 0321. In the latter case, the hybrid's earing date exactly coincided with the earing date of sowings with yarovized seed of the winter parent, Kooperatorka, whose light stage is shorter than that of pure line 0321.

Third Group. We include in the third group the remaining nine combinations in the crossing of winter wheat with spring wheat. The response of the first generation of this group to yarovization is either slight or altogether absent. The hastening effect on some hybrids of this group does not exceed 3 to 5 days. It is quite likely that if the hybrids of this group were cultivated under other conditions than those of our experiments in 1930, and yarovized at other temperatures, yarovization might produce in them, too, a hastened development, just as in the second group. During the year 1936, for example, in Bezenchuk, yarovization hastened the development of the F_1-hybrids of 329×Prelude (experiments of Saprygina). But in our own experiments in 1930, the hybrids of all combinations in this group lagged very much behind the spring wheat Prelude and did not respond to yarovization at all. The general conclusion that we may draw from our experiments with yarovized hybrids may be formulated thus: A short stage of yarovizа-

tion is dominant in hybrids over a long one, but this dominance is relative, not complete.

We pass now to a consideration of the causes of intermediate inheritance of cold-resistance in F_1-hybrids, which is especially plainly shown by hybrids in the combinations of winter wheat × spring wheat. Owing to incomplete dominance in F_1-hybrids of the short stage of yarovization over the long one, in summer and fall sowings the hybrids develop more slowly than their parent spring forms. At the beginning of the paper, we mentioned that the capability in winter wheats of being hardened to low temperatures falls progressively only with the onset of the light stage. The light stage in winter wheat × × spring wheat F_1-hybrids begins later in the fall than in the case of their parent spring forms. In autumn, the length of day quickly decreases, therefore, having entered into the flowering stage later, the hybrids pass through it under circumstances somewhat less favourable for their development than in the case of the spring wheats. Hence, in the fall, hybrids undergo hardening to low temperatures at a younger age and enter the winter more hardened, more resistant to low temperatures than their parent spring forms. The same cause of the relative dominance of the short yarovization stage over the long one produces a lower cold-resistance on the part of winter wheats × spring wheats hybrids in comparison with the parent winter forms.

The length of the light stage of development in wheats has a primary significance for their cold-resistance ([9],[10]). Hybrids from the crossing of one variety of winter wheat with various varieties of spring wheats are the more cold-resistant and approach the nearer to the cold-resistance of the parent winter wheat, in proportion as the light stage of the parent spring wheat is longer ([12]).

If we add here that the development of a long light stage takes place late in the fall, when not only shortened days but also low temperatures retard passage through the stage, then the enormous difference in the cold-resistance of the winter wheats × spring wheats F_1-hybrids will be clear.

In this case, to the causes already noted which retard the developmental stages in the hybrids, must be added the effect of the duration of the light stage. The longer is the light stage in wheats, the longer is that period of time, after passing through stage of yarovisation, which is necessary for attaining that degree of development in the light stage when the threshold of hardening is reached—that is, when the plant can no longer, under any conditions, get sufficiently hardened to low temperatures.

In winter wheats of the northern latitudes both stages of development are long. Such «genuine» winter wheats were used in our crossings. Owing to the dominance of the short light stage of development over the long ([6]), the longer the light stage in the spring wheat that is taken for crossing with a winter wheat, the more slowly will the light stage develop in the F_1-hybrids. If we add to this the fact that the development of this long stage takes place late in the fall, when not only short days but also low temperatures may retard passage through the stage, then the enormous difference in the cold-resistances of F_1-hybrids from the crossing of one variety of winter wheat with varieties of spring wheats that differ greatly in the length of their light stages, will be clear.

We have examined the causes of the intermediate inheritance of cold-resistance in F_1-hybrids of winter wheat × spring wheat in which the significance of the development by stages with regard to cold-resistance is particularly easy to discover. In the case of hybrids of winter wheat × winter wheat, of course, all that has been pointed out is of no less importance than for

the hybrids we have considered, but in this group of hybrids, the material at our disposal is not yet sufficient to give such an analysis as for hybrids of spring wheat × winter wheat.

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Received 13. XII. 1936.

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